Matching Points with Rectangles and Squares

Sergey Bereg, *Nikolaus Mutsanas* & Alexander Wolff

23 January 2006

Outline

- Introduction
  - Matching in graphs and in the plane
  - Previous results
  - Open problems
- Rectangles
  - General position
  - 1/2-Approximation
  - 4/7-Approximation
- Squares
  - Is there a strong realization?
  - Application to map labeling
  - NP-completeness

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Matching algorithms

Maximum Matching in graphs
[Micali & Vazirani '80] $O(\sqrt{n}m)$

Euclidean Minimum-Weight Perfect Matching
(matching points with line segments of minimum total length)
[Vaidya '88] $O(n^{2.5}\log^4 n)$
[Varadarajan & Agarwal '99] $O((n/\varepsilon^3)\log^6 n)$

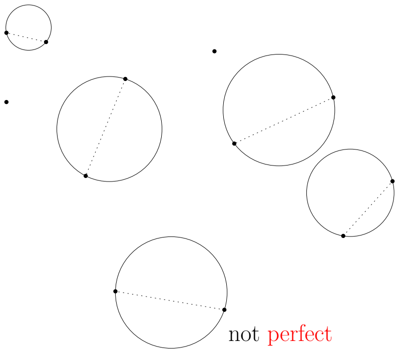
Matching with segments, rectangles, squares, disks...

Matching in the plane

Definition

- Matching is perfect: covers all points.
- Matching is strong: no overlap.

Matching in the plane



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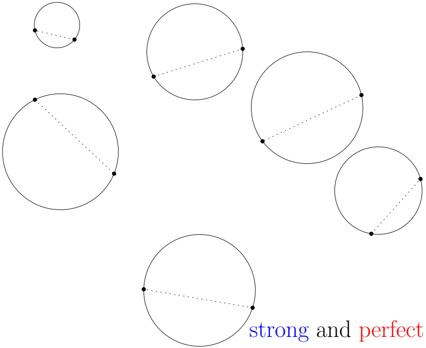

Definition

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Already known results

Let P be a set of $2n$ points in the plane.

Theorem (Rendl & Woeginger, '93)

It is NP-hard to decide whether P admits a strong rectilinear segment matching.

Theorem (Ábrego et al. '04)

If P is in general position (no two points on a horiz./vert. line), then P admits

- *a perfect* disk *matching and a perfect* square *matching.*
- *a strong* disk *matching covering at least 25% of P.*
- *a strong* square *matching covering at least 40% of P.*

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Open Problems

Questions

– How many points can be matched strongly?
– Does a given matching have a strong realization?

	matching size	ex. strong realization?
segments	100%	$O(n \log n)$
rectangles	100%	$O(n \log n)$
squares	40%	?
disks	25%	?

Points in general position

Open Problems

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disks	25% / ?	?

Points in general position / General point sets

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Questions

– How many points can be matched strongly?
– Does a given matching have a strong realization?

	matching size	ex. strong realization?
segments	100%	$O(n \log n)$
rectangles	100% / 57%	$O(n \log n)$
squares	40% / ?	? / $O(n^2 \log n)$
disks	25% / ?	?

Points in general position / General point sets

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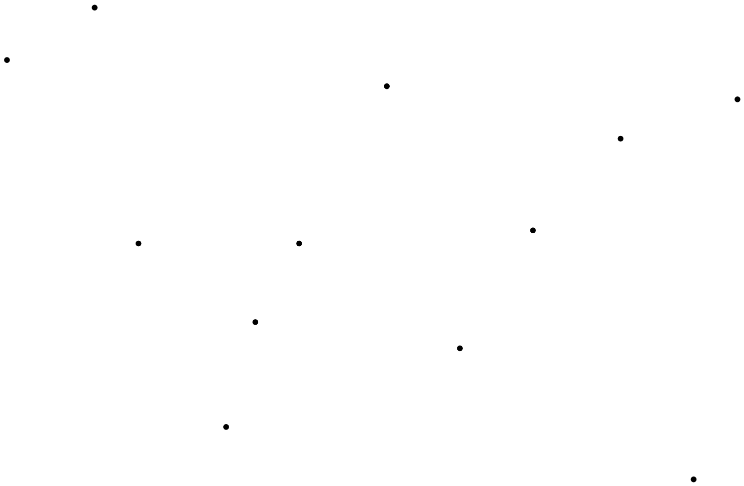
General position



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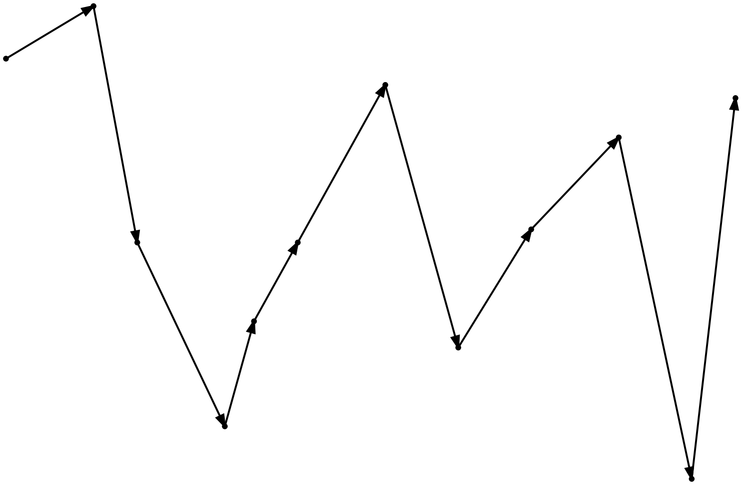

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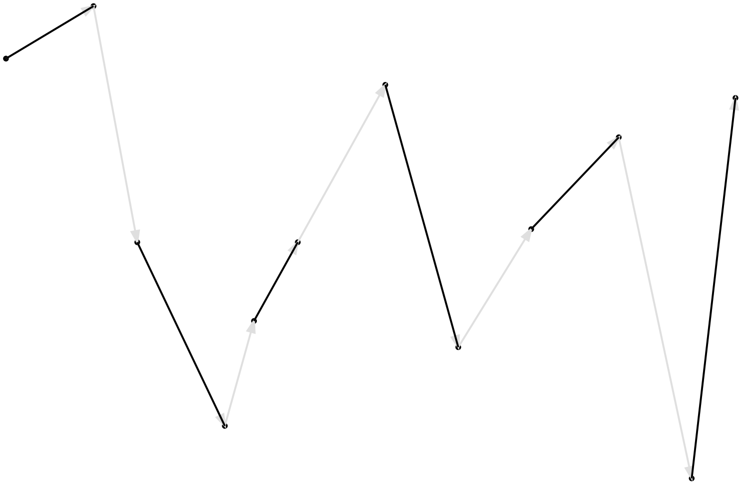

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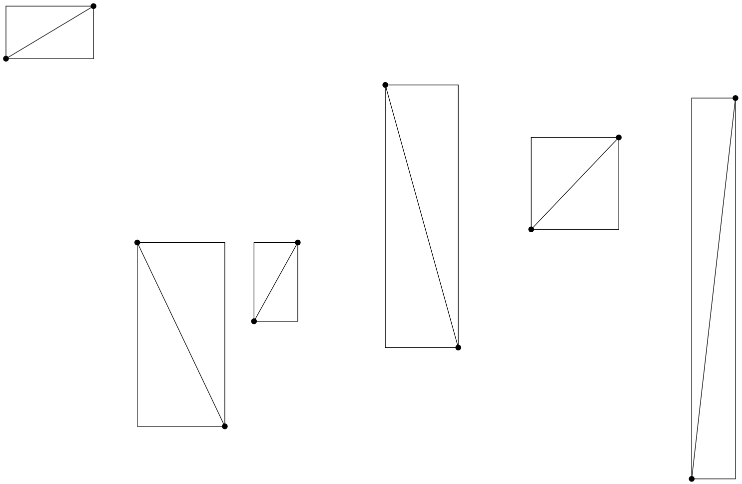

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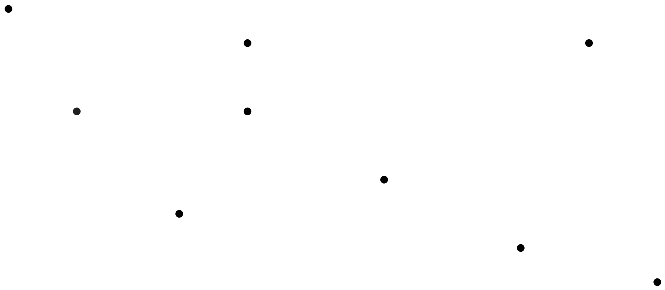

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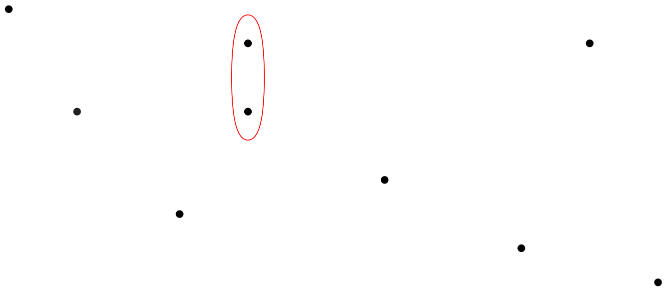

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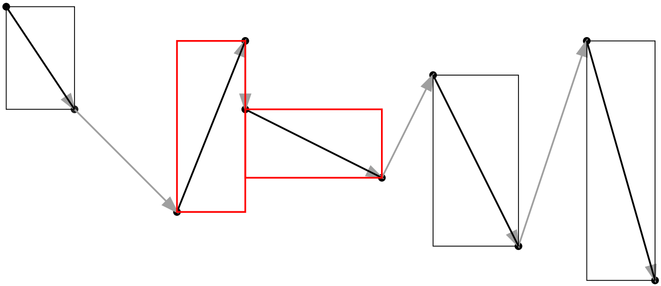
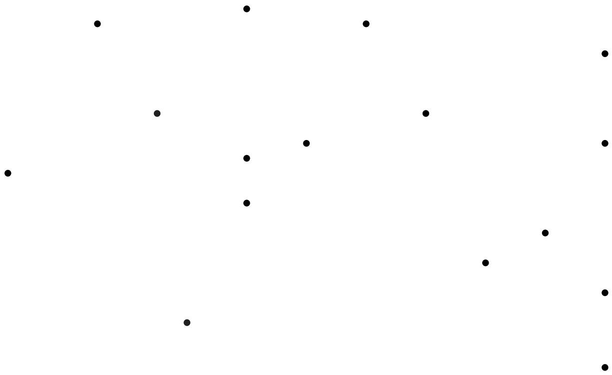


1/2-Approximation

Divide into subsets → match subsets → join



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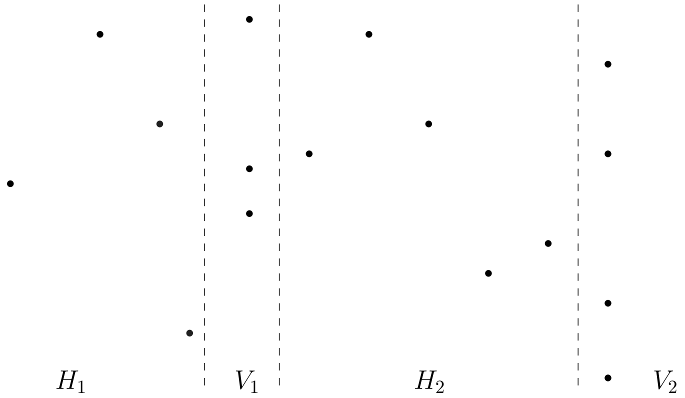

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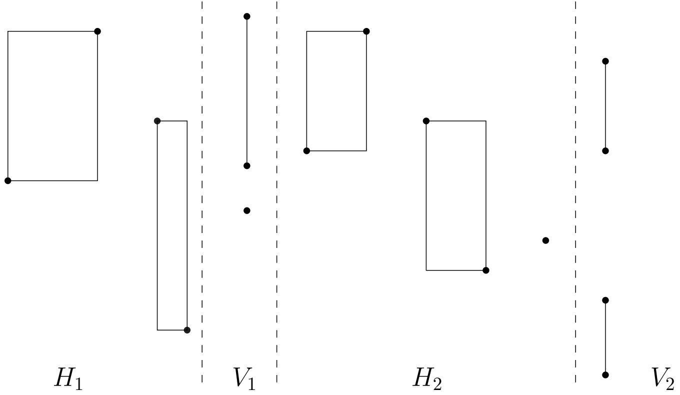

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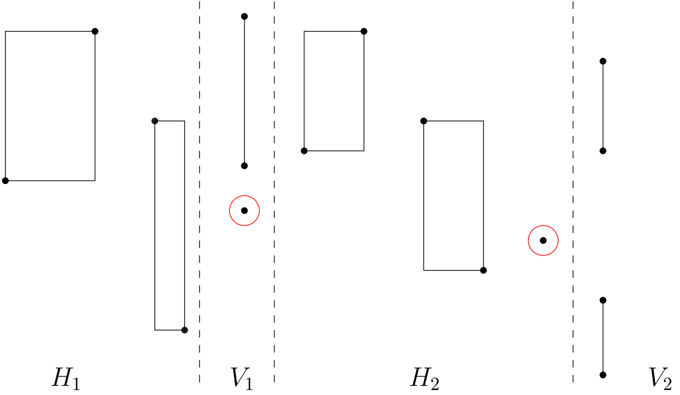

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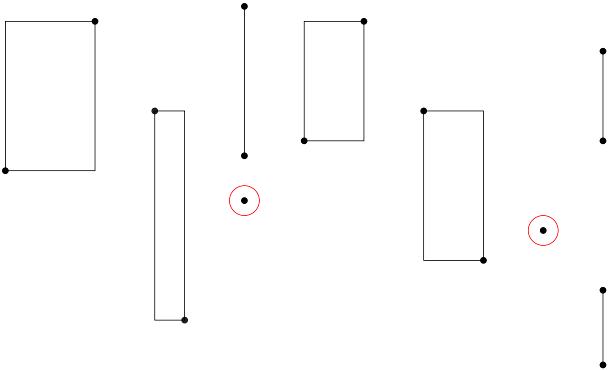

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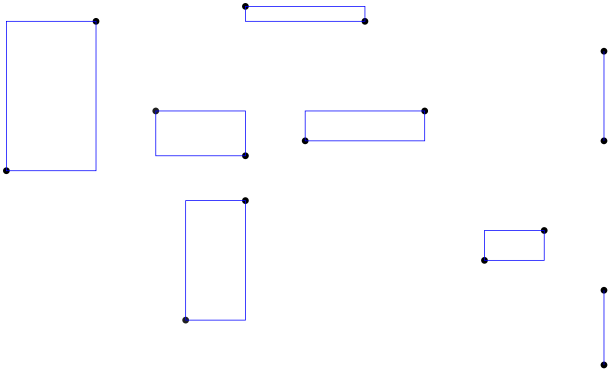
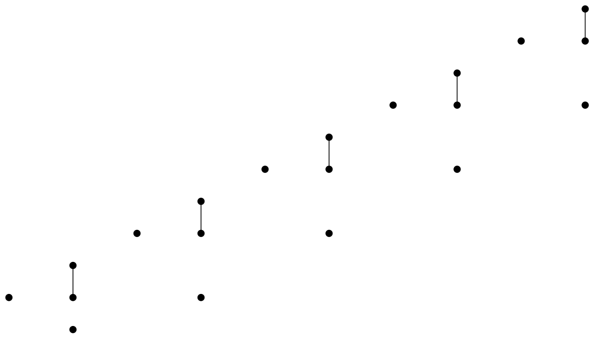


1/2-Approximation - worst case

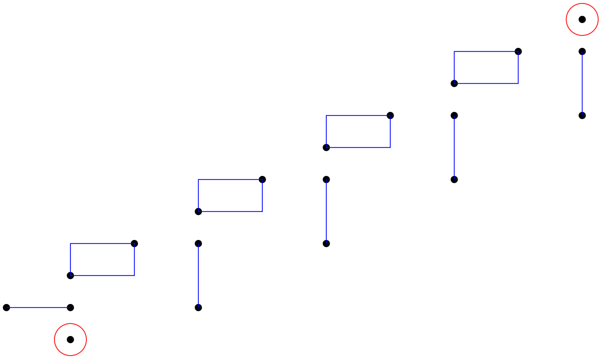
Worst Case



Matching with $n/2$ points.

1/2-Approximation - worst case

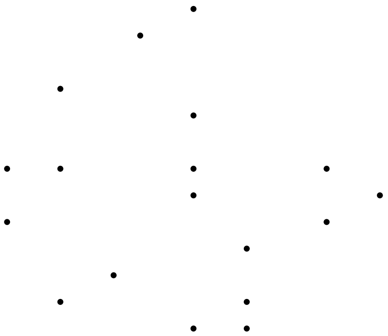
Worst Case



Optimal matching with $n-2$ points.

4/7-Approximation

Basic Idea:



For an arbitrary point set P

- Partition P into subsets
- Match at least $4/7$ of the points in each subset
- Overall matching covers at least $4/7$ of P

4/7-Approximation

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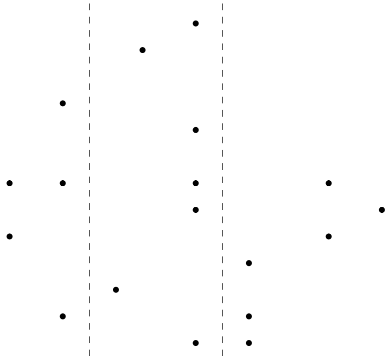

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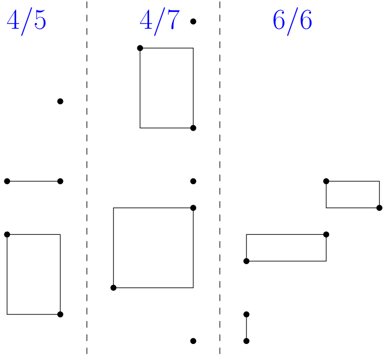

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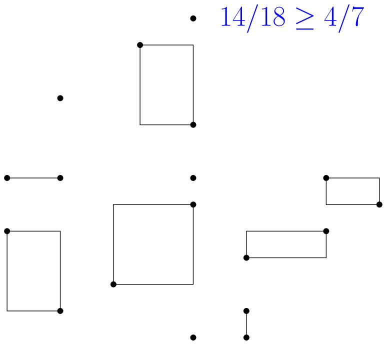

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4/7-Approximation

v_1 even $\quad v_1 = 1 \quad v_1 \geq 3$, odd

v_1 even

4/7-Approximation

v_1 even $\quad v_1 = 1 \quad v_1 \geq 3$, odd

$6/6$ $\quad v_1$ even

$\rightarrow$ match all points

4/7-Approximation

v_1 even $\quad v_1 = 1 \quad v_1 \geq 3$, odd

-
-
-

$v_1 \geq 3$, odd

4/7-Approximation

v_1 even $\quad$ $v_1 = 1$ $\quad$ $v_1 \geq 3$, odd

$2/3$ $\quad$ $v_1 \geq 3$, odd

$\rightarrow$ match all but one point

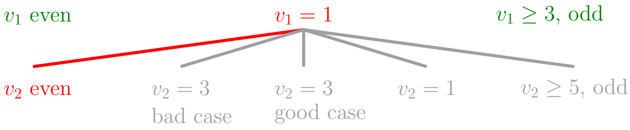
4/7-Approximation

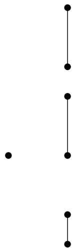
v_1 even $\quad v_1 = 1 \quad v_1 \geq 3$, odd

$$v_1 = 1$$

-

4/7-Approximation

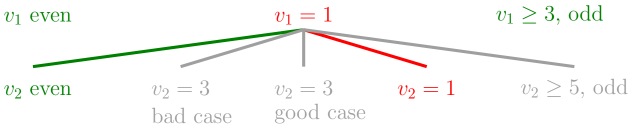


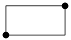


$6/7$ $\quad v_1 = 1,\ v_2$ even

$\rightarrow$ match v_2

4/7-Approximation

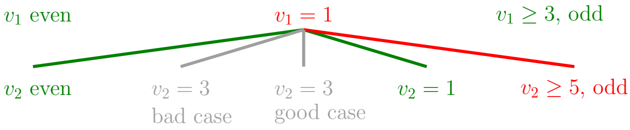


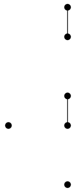


$2/2$ $\quad v_1 = 1, \; v_2 = 1$

$\rightarrow$ match v_1 to v_2

4/7-Approximation

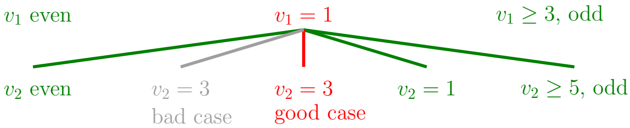


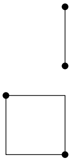


$4/6$ $\quad v_1 = 1,\ v_2 \geq 5$, odd

$\rightarrow$ match v_2

4/7-Approximation

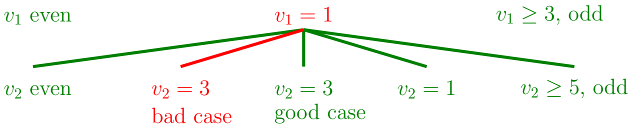


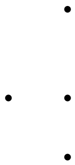


$4/4$ $\quad v_1 = 1, v_2 = 3$ (good)

$\rightarrow$ match all points

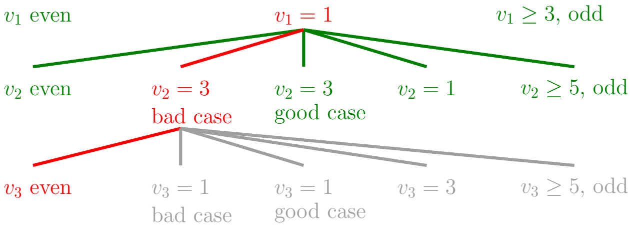
4/7-Approximation

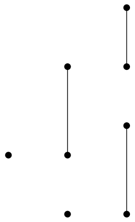




$v_1 = 1, v_2 = 3$ (bad)

4/7-Approximation

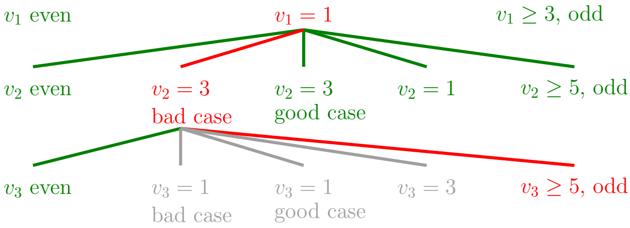


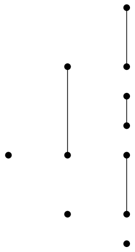


$6/8$ $\quad v_1 = 1, v_2 = 3$ (bad), v_3 even

$\rightarrow$ match v_2 and v_3

4/7-Approximation

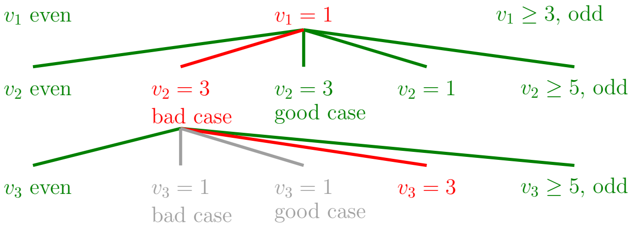


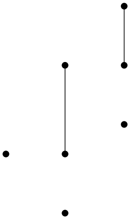


$8/11$ $\quad v_1 = 1, v_2 = 3$ (bad), $v_3 \geq 5$ odd

$\rightarrow$ match v_2 and v_3

4/7-Approximation

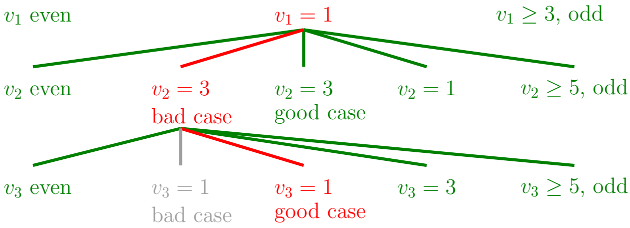


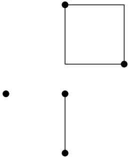


$4/7$ $\quad v_1 = 1, v_2 = 3$ (bad), $v_3 = 3$

$\rightarrow$ match v_2 and v_3

4/7-Approximation



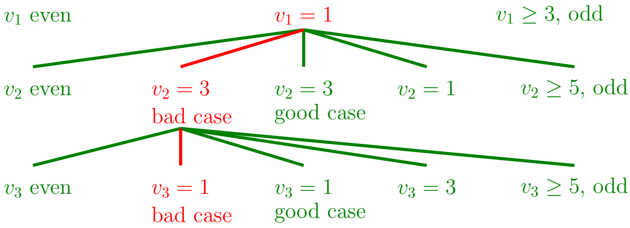


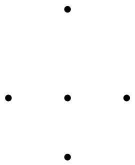
$4/5$

$v_1 = 1, v_2 = 3$ (bad), $v_3 = 1$ (good)

$\rightarrow$ match v_2 with v_3

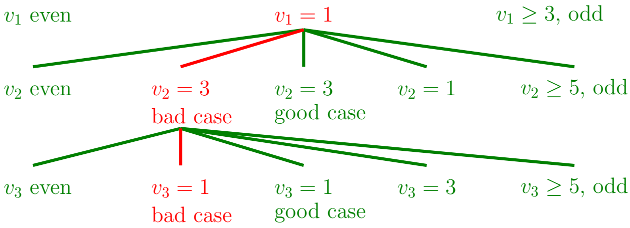
4/7-Approximation

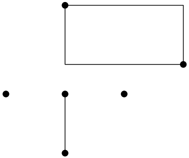




$v_1 = 1$, $v_2 = 3$ (bad), $v_3 = 1$ (bad)

4/7-Approximation



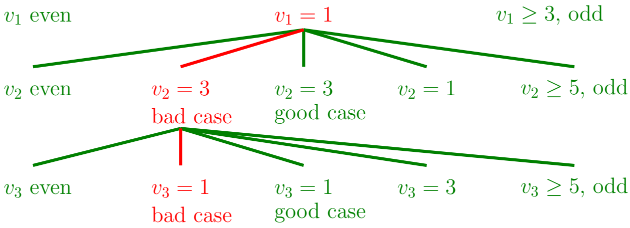


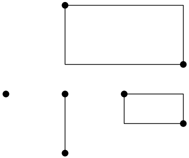
$4/6$

$v_1 = 1$, $v_2 = 3$ (bad), $v_3 = 1$ (bad)

$\rightarrow$ use v_4

4/7-Approximation





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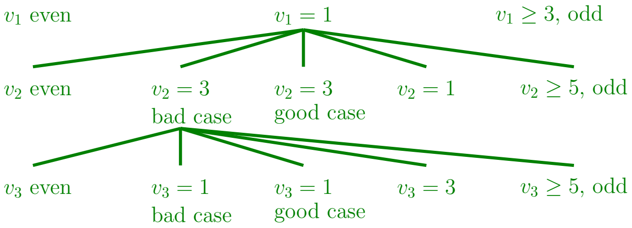

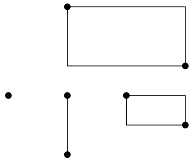

$4/6$

Theorem (Bereg, Mutsanas & Wolff, '05)

In any set P of n points, $\geq 4/7 \cdot n - 5$ points can be matched with rectangles in $O(n \log n)$ time.

But...

There are point sets, for which $\leq 2\lfloor n/3 \rfloor$ points can be matched!

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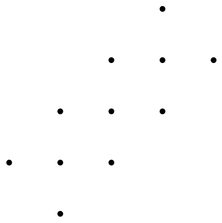

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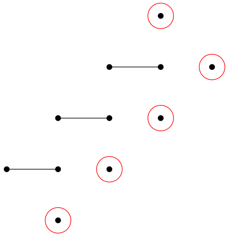
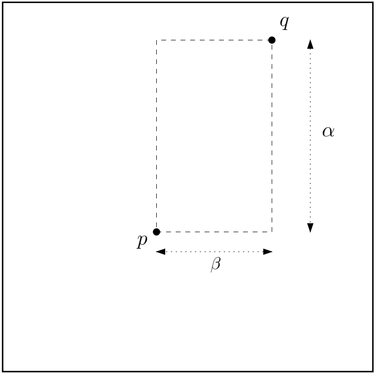


Outline

- Introduction
  - Matching in graphs and in the plane
  - Previous results
  - Open problems
- Rectangles
  - General position
  - 1/2-Approximation
  - 4/7-Approximation
- Squares
  - Is there a strong realization?
  - Application to map labeling
  - NP-completeness

Minimal squares

Minimal squares: points lie on the boundary.



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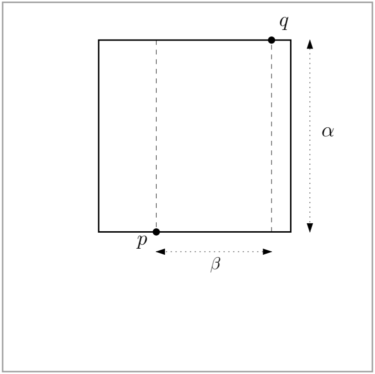
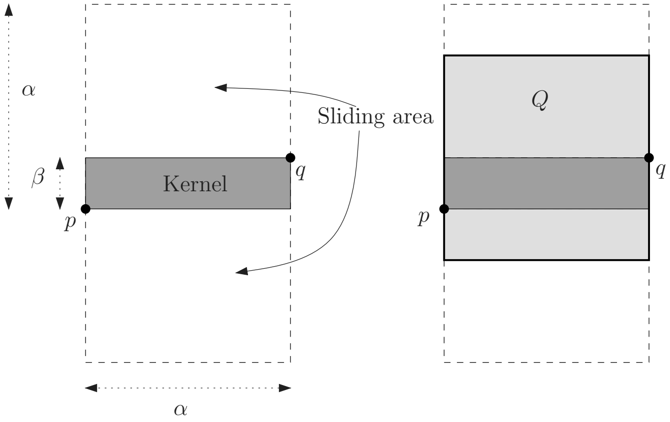
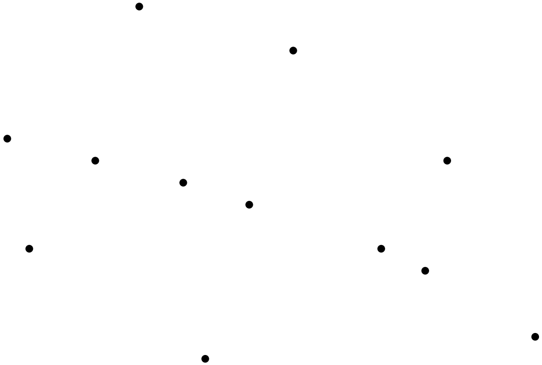


Sliding squares



Is there a strong realization?



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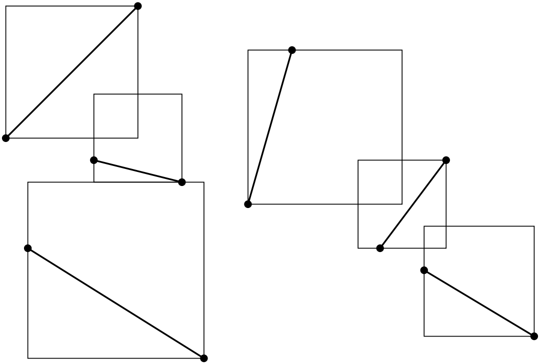

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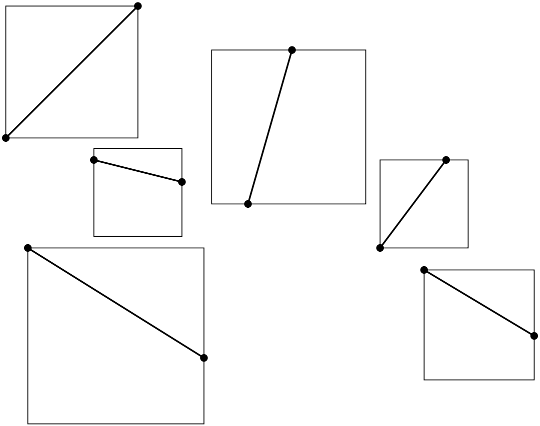

Help from map labeling

Labeling rectilinear segments

Given: Set of rectilinear segments, $B \in \mathbb{R}$.
Question: Is there a labeling of height B?

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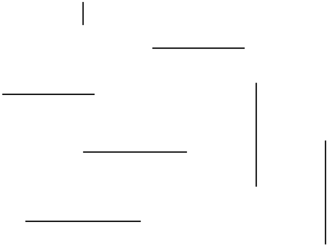

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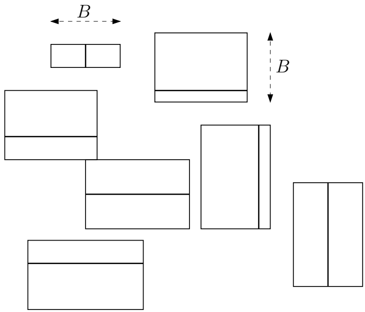

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Labeling rectilinear segments

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Theorem (Kim, Shin & Yang, '99)

Rectilinear segment labeling is solvable in $O(n^2 \log n)$ time.

Squares - canonical form

Let squares slide

- for *vertical* kernels *leftwards* as far as possible.
- for *horizontal* kernels *downwards* as far as possible.

When does a square stop sliding?

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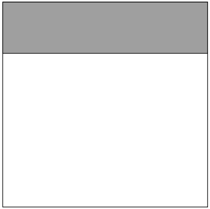

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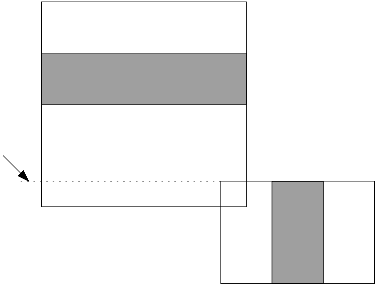

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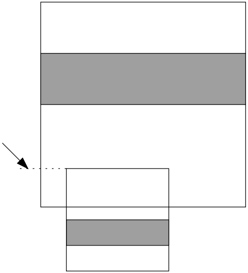

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Observations

– The resulting positions can be computed in advance.
– Every square has $O(n)$ relevant positions.

Squares - Decision Algorithm

Problem

Given: $P \subseteq \mathbb{R}^2$, matching $M \subseteq \binom{P}{2}$
Question: Is there a strong square realization of M?

- Do kernels overlap?
- Calculate relevant positions.
- Solve decision problem with 2-SAT.

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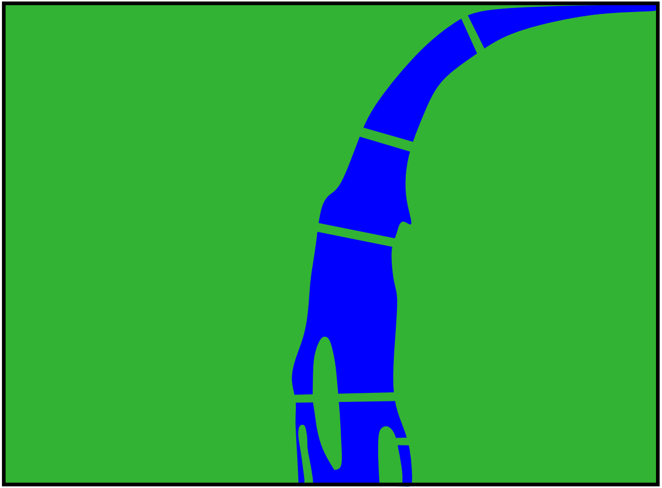
Conclusion

The decision problem can be solved in $O(n^2 \log n)$ time.

Labeling points with sliding labels



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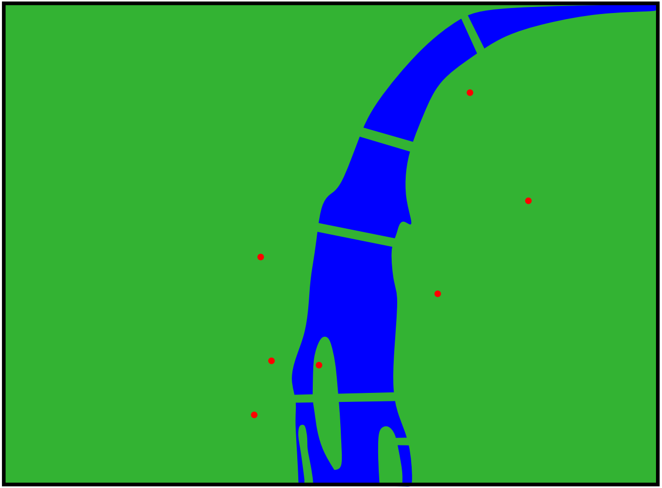

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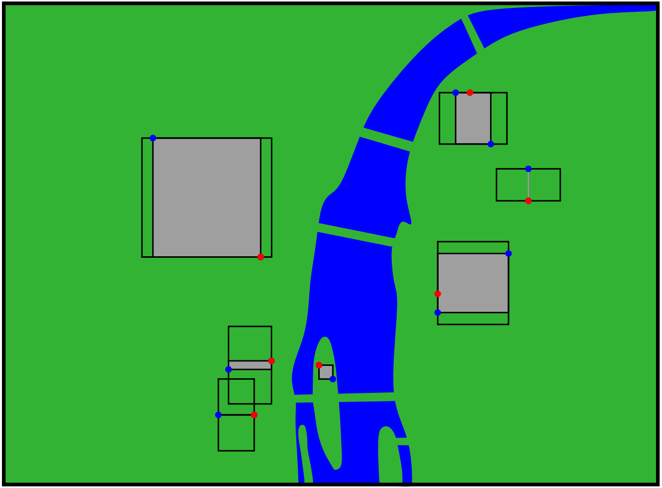

NP-completeness

ESPSM

Given: Point set $P \subseteq \mathbb{R}^2$

Question: Does a strong perfect square-matching exist?

Theorem (Bereg, Mutsanas & Wolff '05)

ESPSM is NP-hard.

Proof.

By reduction from PLANAR 3-SAT to ESPSM. □

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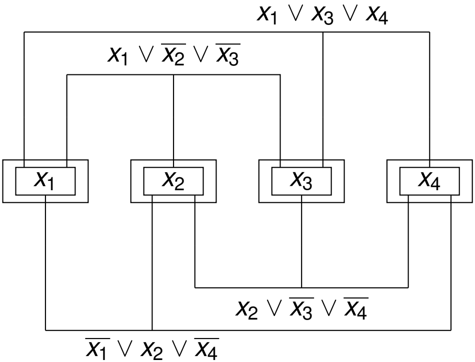
Outline of the Reduction



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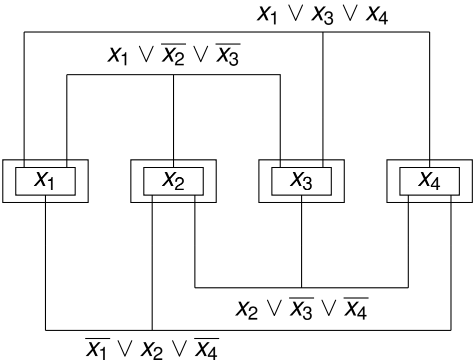

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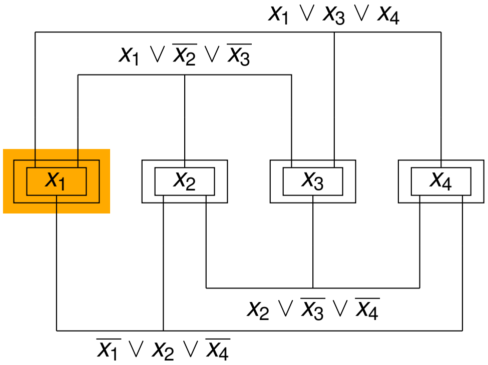

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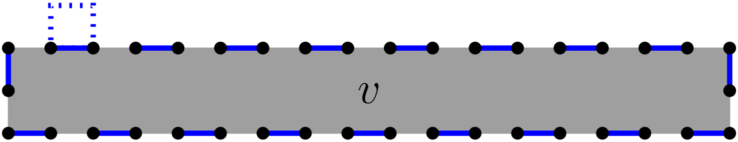

$v =$ true

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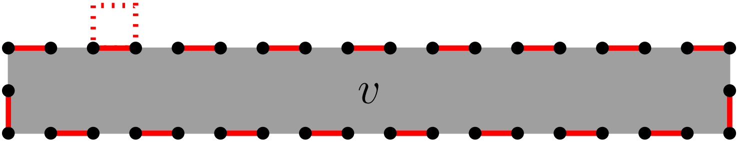

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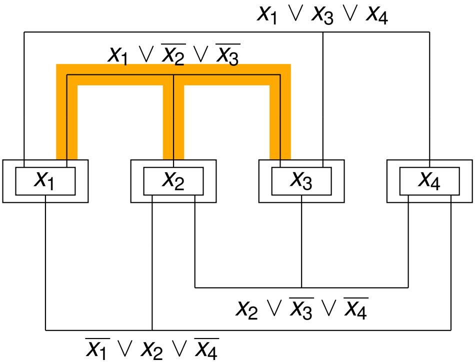

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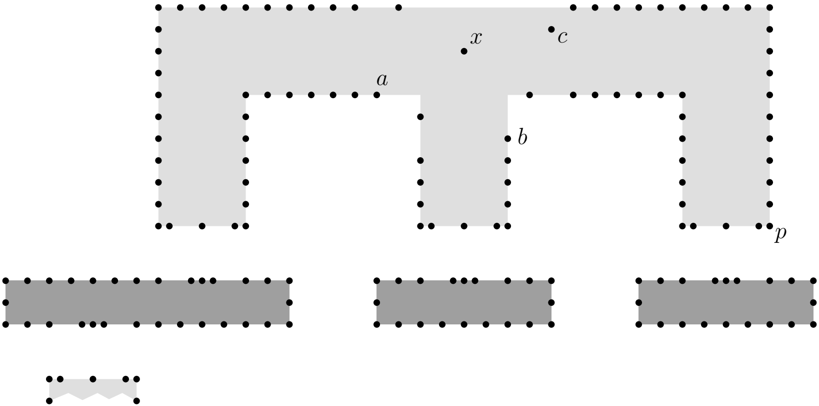

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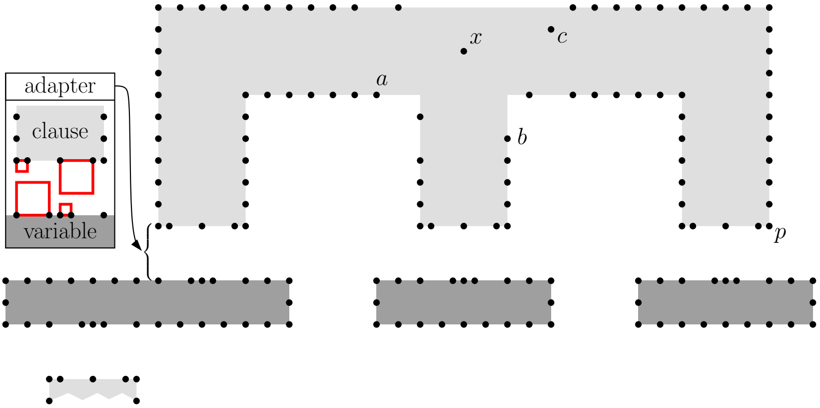

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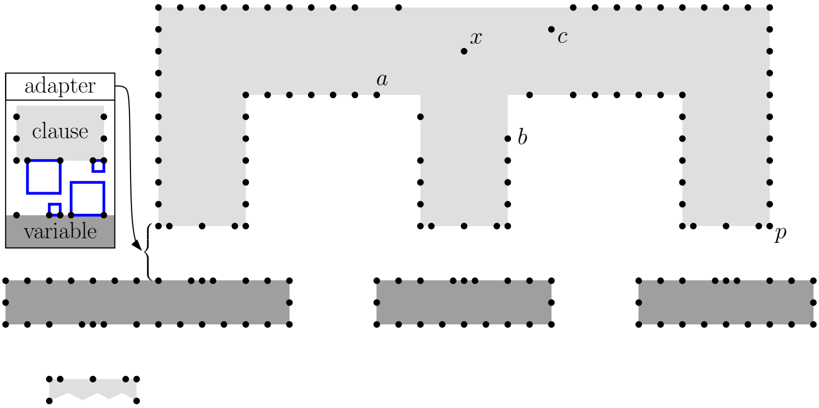

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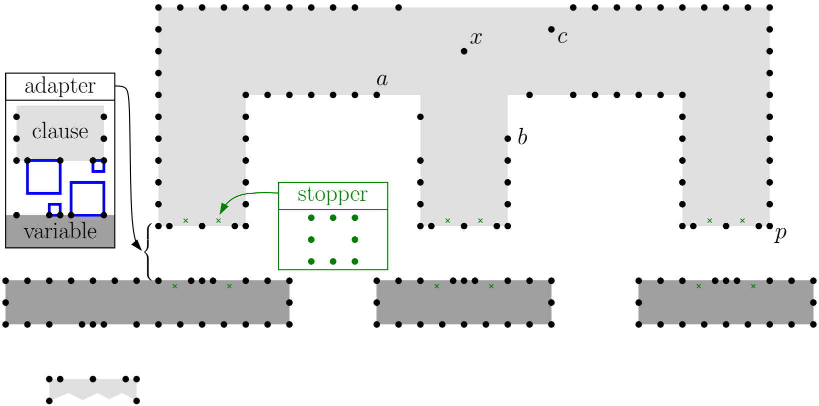

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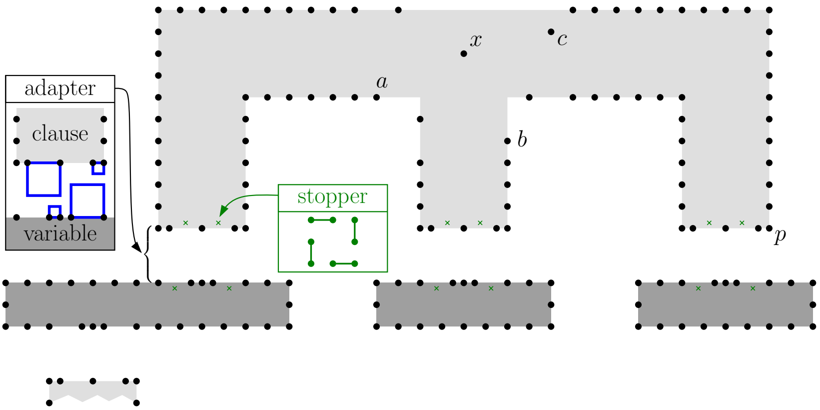

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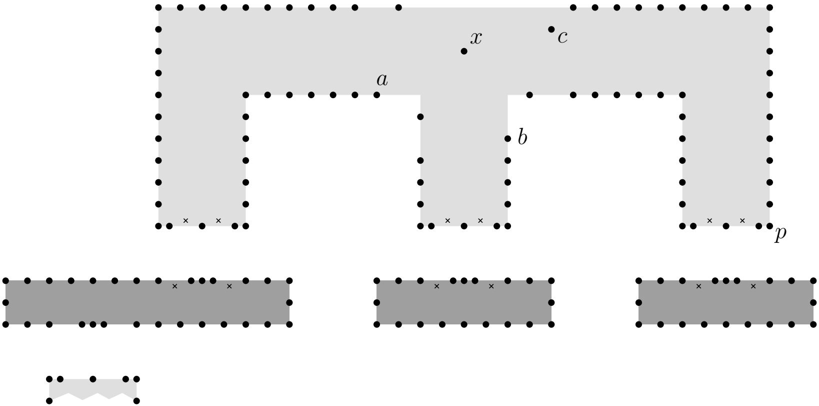

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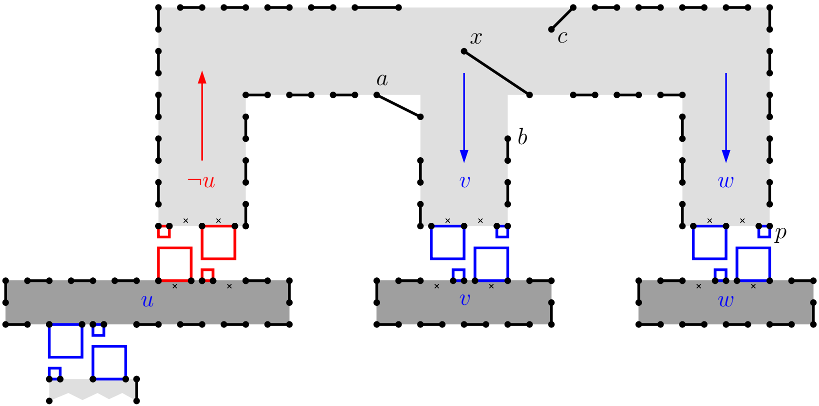

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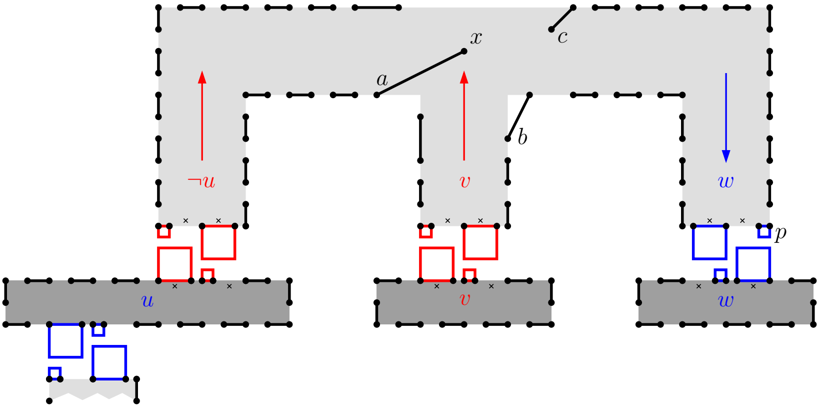

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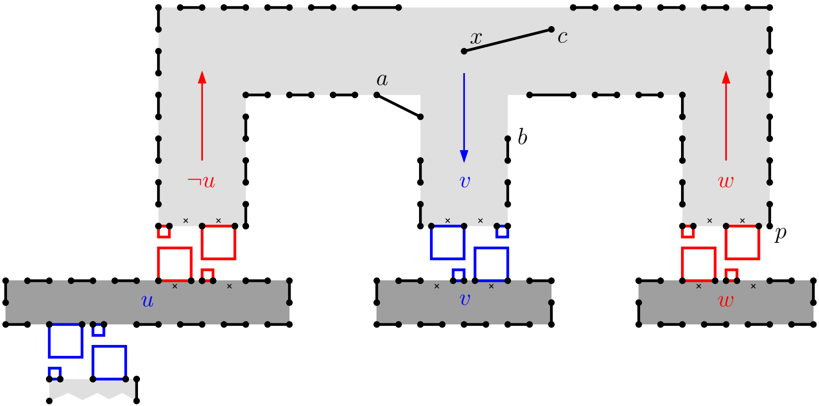

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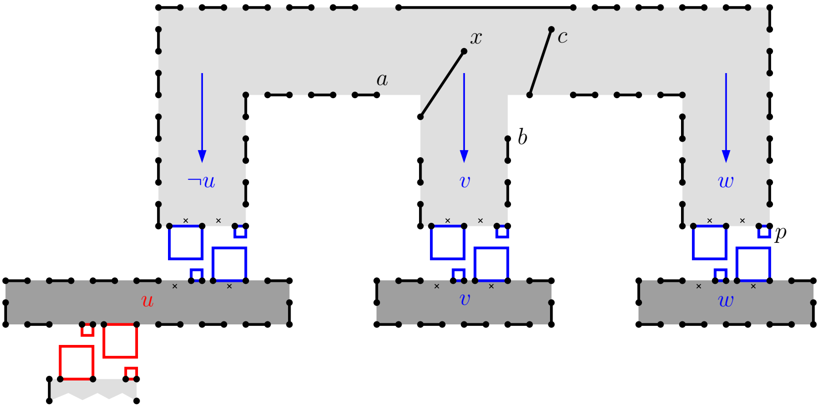

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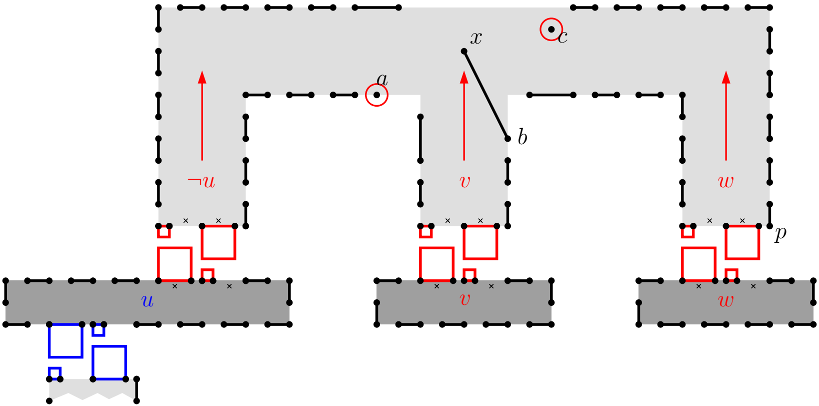

Conclusions

- Upper bound for rectangle-matching $2/3 \cdot \#$ points
- With rectangles we can match $\geq 4/7 \cdot \#$ points
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- Is there a perfect strong square-matching? NP-hard

Open questions

- Match $2/3$ of the points with rectangles? (solved!)
- Approximation algorithms?
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