Searching Paths of Constant Bandwidth

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Overview

- Longest path
- Paths of constant bandwidth
- Fixed-Parameter Tractability
- Conclusions and Open Questions

$\boxed{\text{LONGEST PATH}}$

Given: graph G, integer k

Question: Is there a simple path of length k in G?

LONGEST PATH is NP-complete [GJ97]

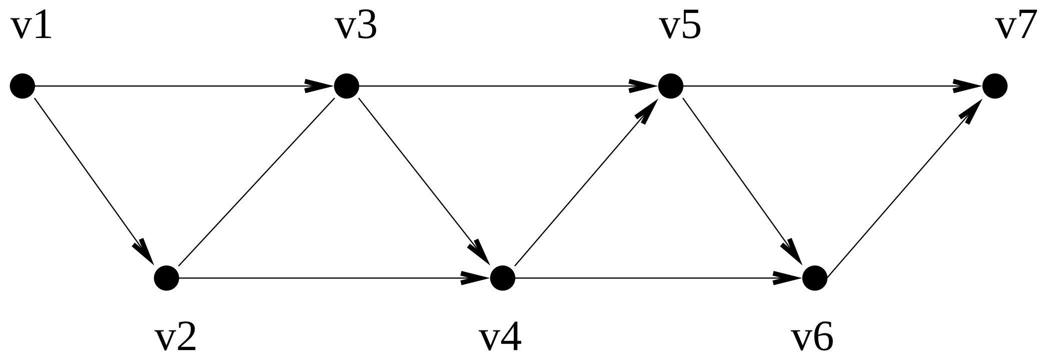
[Monien85]: LONGEST PATH is fixed-parameter tractable in the parameter k.

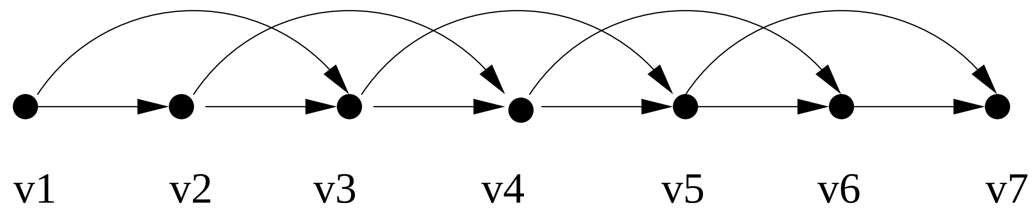
[AlonYusterZwick95]: improved running time using randomization techniques.

Paths of constant bandwidth

A **path of bandwidth w and length k** ((w,k)-*path*) in G is a sequence of $k+w$ vertices $(v_1,\ldots,v_{k+w})$
such that for every i with $1 \leq i \leq k$ and every j with $1 \leq j \leq w$
the pair (v_i, v_{i+j}) is an edge of G.

Two drawings of the same (2,5)-path:





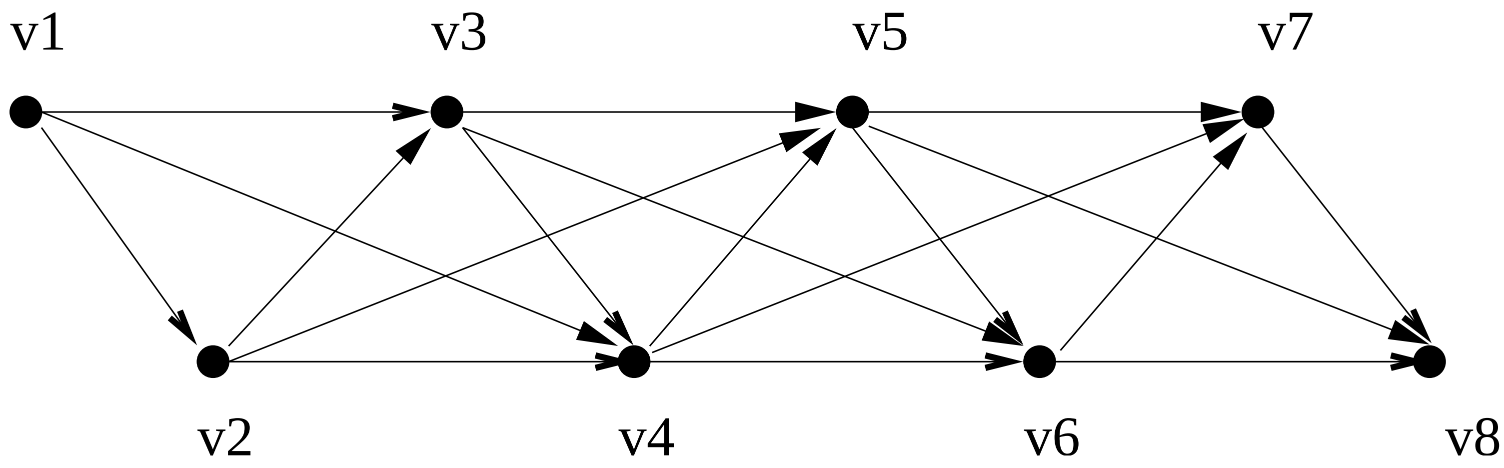
A (w,k)-path $(v_1,...,v_{k+w})$ is

- vertex-disjoint if all v_i are different from each other,

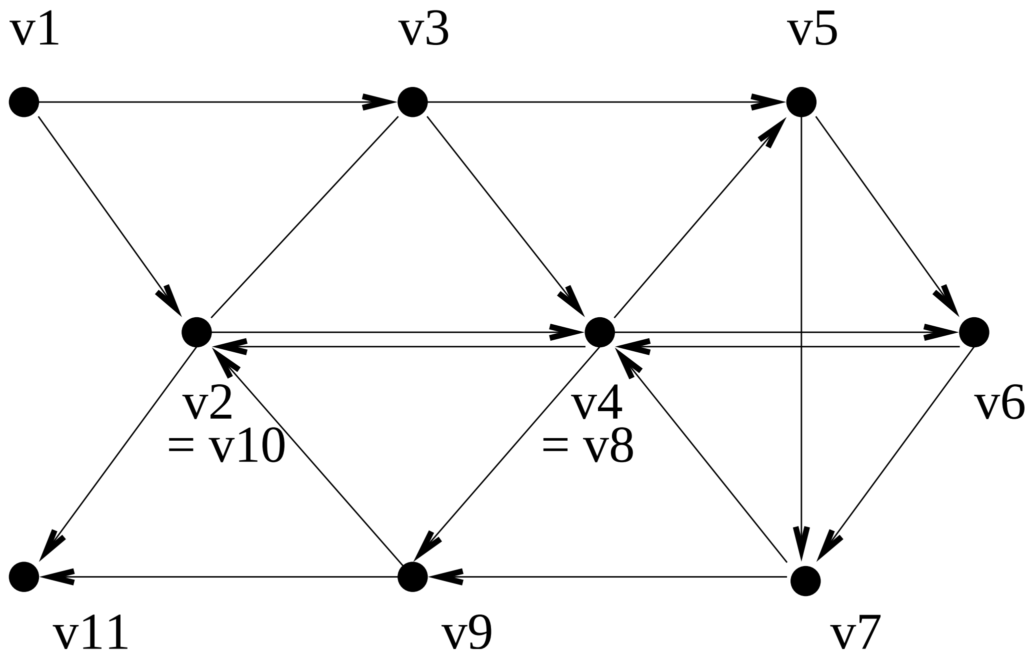
- simple if all k w-tupels $(v_1,...,v_w)$, $(v_2,...,v_{w+1})$, . . . , $(v_k,...,v_{k+w})$ are different from each other.

- deterministic in G if for every $1 \leq i \leq k$, v_{i+w} is the only vertex in the graph G having the property that all edges $(v_i,v_{i+w}),\ldots,(v_{i+w-1},v_{i+w})$ are edges of the graph.

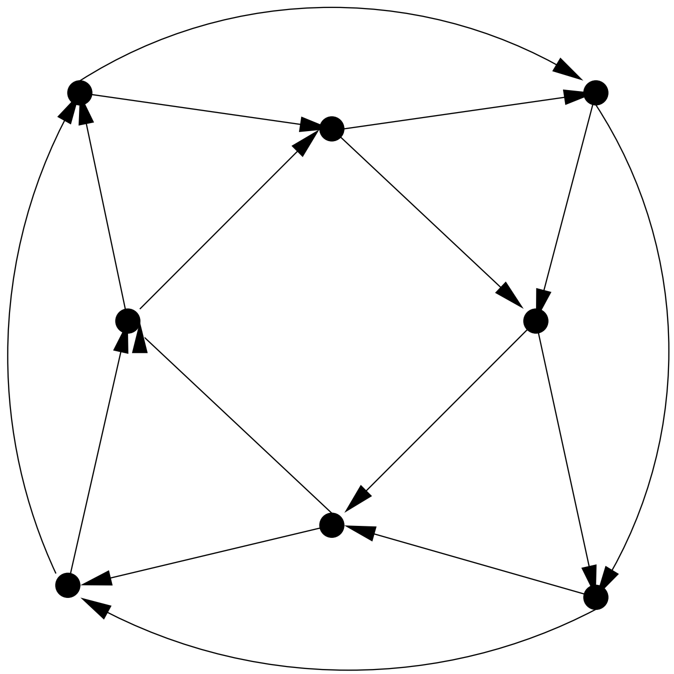
A vertex-disjoint and deterministic (3,5)-path:



A (2,10)-path, deterministic and simple but **not** vertex-disjoint:



A (w,k)-path $(v_1,...,v_{k+w})$ is **a (w,k)-cycle**, if $(v_{k+1}, \ldots, v_{k+w}) = (v_1,\ldots,v_w)$. A (2,8)-cycle, deterministic and vertex-disjoint:



BANDWIDTH-w-PATH: set of pairs $\langle G, k\rangle$ such that G contains a simple (w, k)-path. BANDWIDTH-1-PATH = LONGEST-PATH.

Variations: **UNDIRECTED-** and **DISJOINT-** ...**-CYCLE**

Proposition 1 *For every* $w \geq 1$ *the problem* BANDWIDTH-w-PATH *is* NP*-complete, likewise its variations.*

BANDWIDTH-PATH: set of triples $\langle G, w, k\rangle$ such that G contains a simple (w, k)-path. **PSPACE-complete**

$L \in \mathrm{NP}$ iff there exists a polynomial p and a PTIME computable language C such that $x \in L \iff \exists \mathbf{y} \leq \mathbf{p}(|\mathbf{x}|) : \langle x, y \rangle \in C$.

Witnesses for a path: $(v_1, \ldots, v_{k+w})$

Size of the witnesses: $(k+w)\log(n)$

Fixed-Parameter Tractability

A computational problem consisting of pairs $\langle x, \mathbf{k} \rangle$ is **fixed-parameter tractable in the parameter** k if there is a deciding algorithm for it having run-time $\mathbf{f}(\mathbf{k}) \cdot |\mathbf{x}|^{\mathbf{c}}$ for some recursive function f and some constant c.

[CaiChenDowneyFellows95]: A language $L \in \mathrm{NP}$ consisting of pairs $\langle x, \mathbf{k} \rangle$ is fixed-parameter tractable in the parameter k iff there exists a recursive function $s(k)$ and a PTIME computable language C such that $\langle x, \mathbf{k} \rangle \in L \iff \exists \mathbf{y} \leq \mathbf{s}(\mathbf{k}) : \langle x, \mathbf{k}, y \rangle \in C$.

Theorem 1 *For every* $w \geq 1$ *the problem* **BANDWIDTH-w-PATH** *is fixed parameter tractable in the parameter* k*, likewise its variations. More specifically, there exists an* FPT *guess and check protocol for it with a witness size function* $s(k) = \binom{k}{2} \cdot \log k$ *and a witness checker having runtime* $O(w \cdot k^2 \cdot |E|^w \cdot |V|^w)$.

Idea (Example $w = 1$): If there is a path, there is an witness to construct the lexicographically smallest path.

In the i-th step the algorithm tries to construct for every vertex v a path of length i ending with v.

Step $i = 2$: Start with the smallest predecessor u of v, use $k-1$ numbers $\in \{0, 1\}$ from the witness.

Advice 1: u is needed later, try next u.

General Step i: Start with the smallest predecessor u of v, use $k-i+1$ numbers $\in \{0, i-1\}$ from the advice.

Advice $j > 0$: the j-th vertex in the path to u is needed later, try next u consistent with previous information (keep list of at most $k-1$ vertices needed later).

General case $w > 1$: Use w-tuples of vertices.

Conclusions and Open Questions

We obtained fixed-parameter tractability of BANDWIDTH-w-PATH by presenting an FPT guess and check protocol with a witness size function of $\binom{k}{2} \log k$.

Question: Can this be improved to some **quasi-linear** function?

Thank you for your attention.