The Complexity of Problems on Implicitly Represented Inputs

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Contents

Implicit Data Representation

Definition

An implicit data representation avoids explicit enumeration of single elements.

- Focus: Representation by Boolean functions.
- Consider string $I \in \{0,1\}^n$ of length $n = 2^m$.
- Define the characteristic function $\chi_I : \{0,1\}^m \to \{0,1\}$ of I by

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for $x \in \{0,1\}^m$.

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Implicit Algorithms and OBDDs

- Popular data structure for Boolean functions:
  Ordered Binary Decision Diagrams (OBDDs)
- OBDDs of structured functions are known to be very succinct.
- OBDDs offer efficient algorithms for functional operations.
- $\Rightarrow$ Implicit/symbolic algorithms:
  - Represent input $I \in \{0,1\}^n$ implicitly as OBDD of χ_I.
  - Compute OBDD of χ_O for output $O \in \{0,1\}^*$.
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- OBDD $\mathcal{G}_f$ is acyclic digraph having inner nodes and sinks.
- Inner nodes: Variable label, 0- and 1-edge
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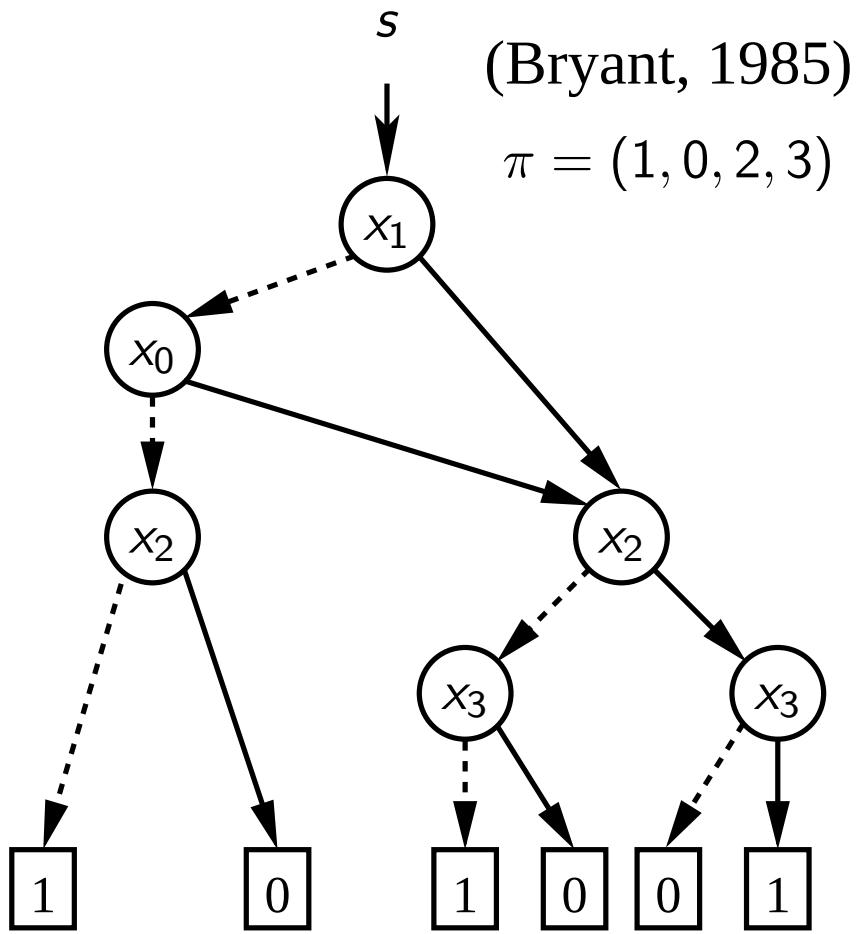

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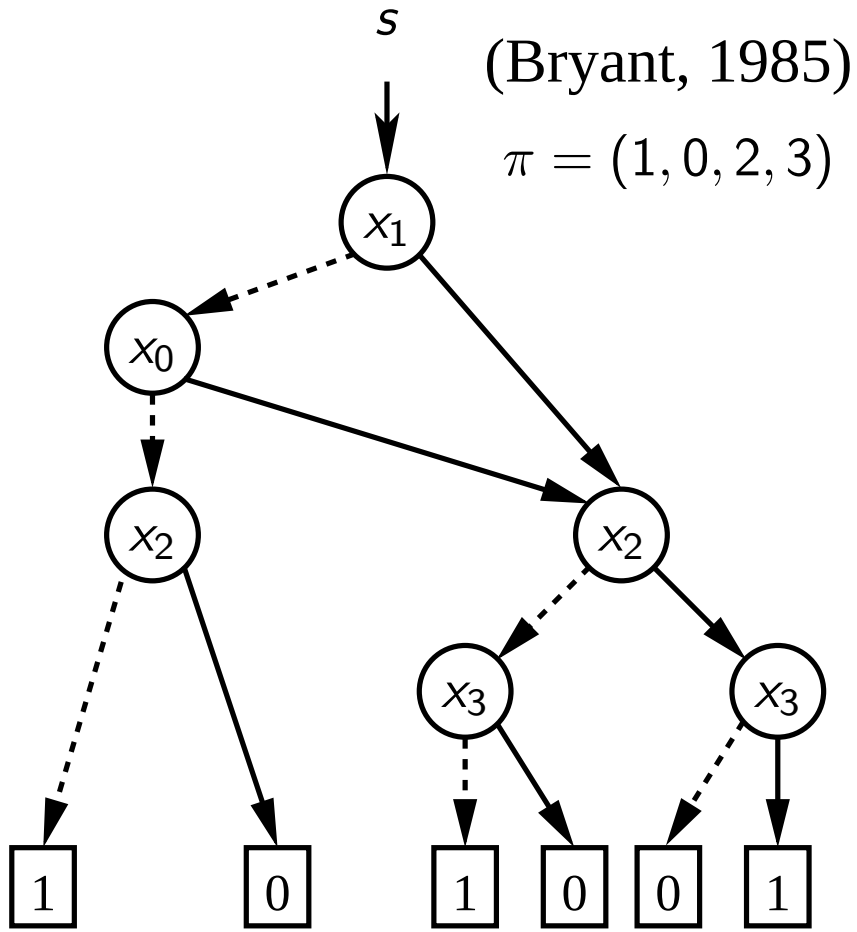

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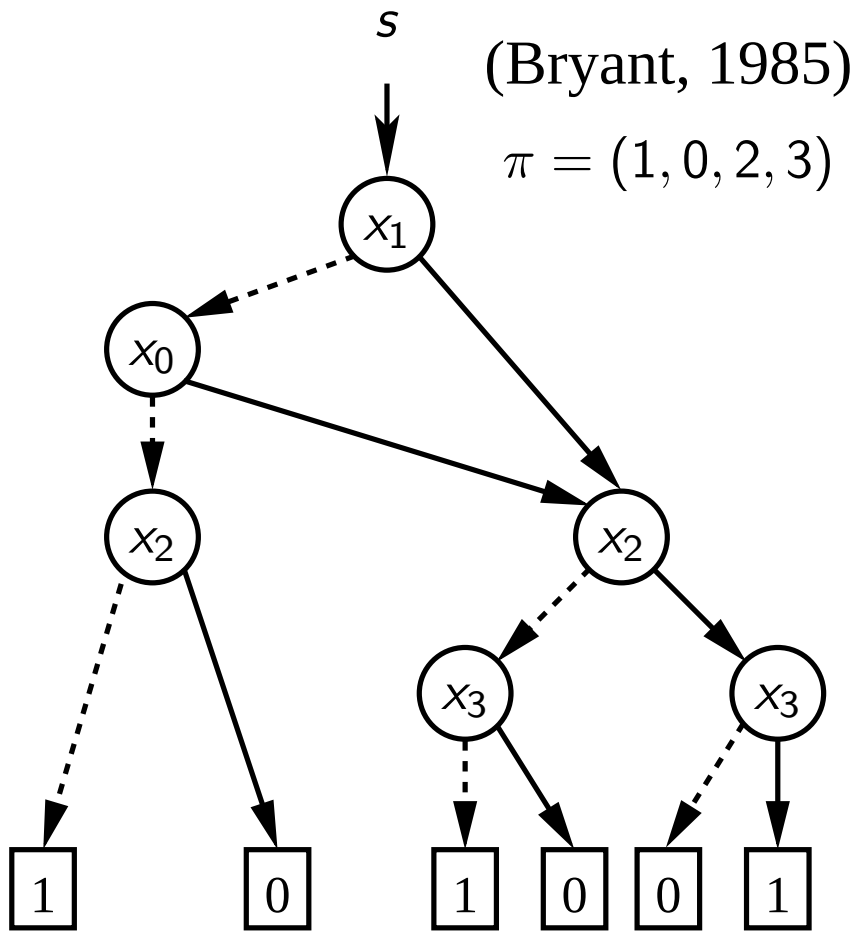

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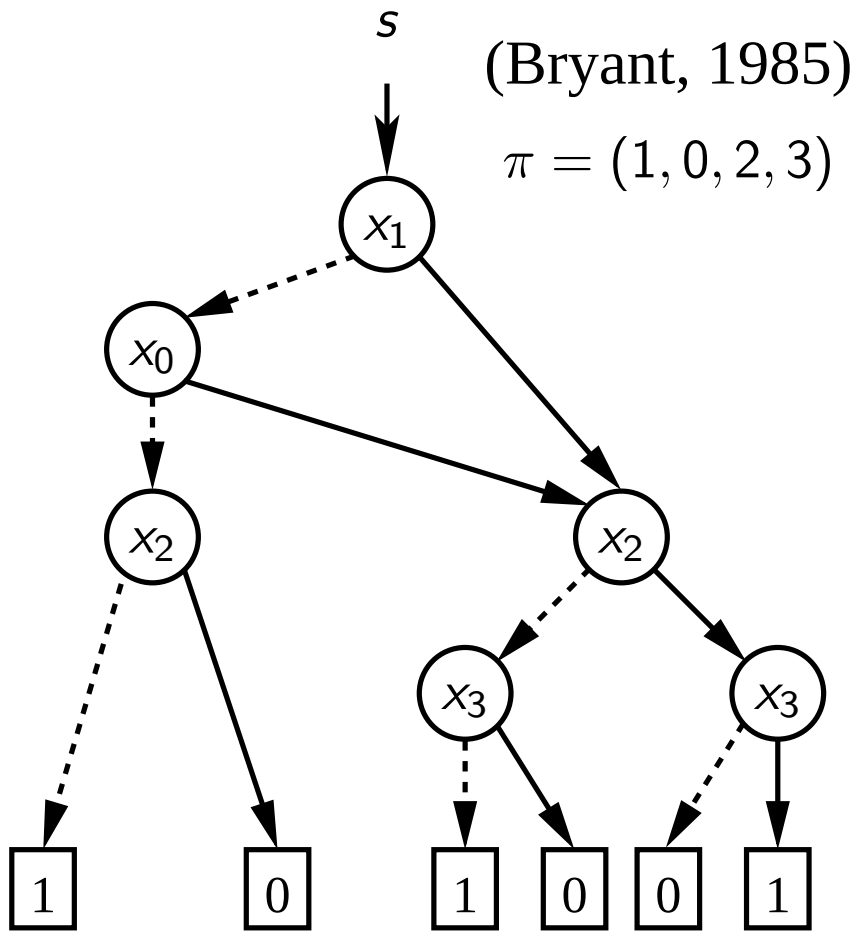

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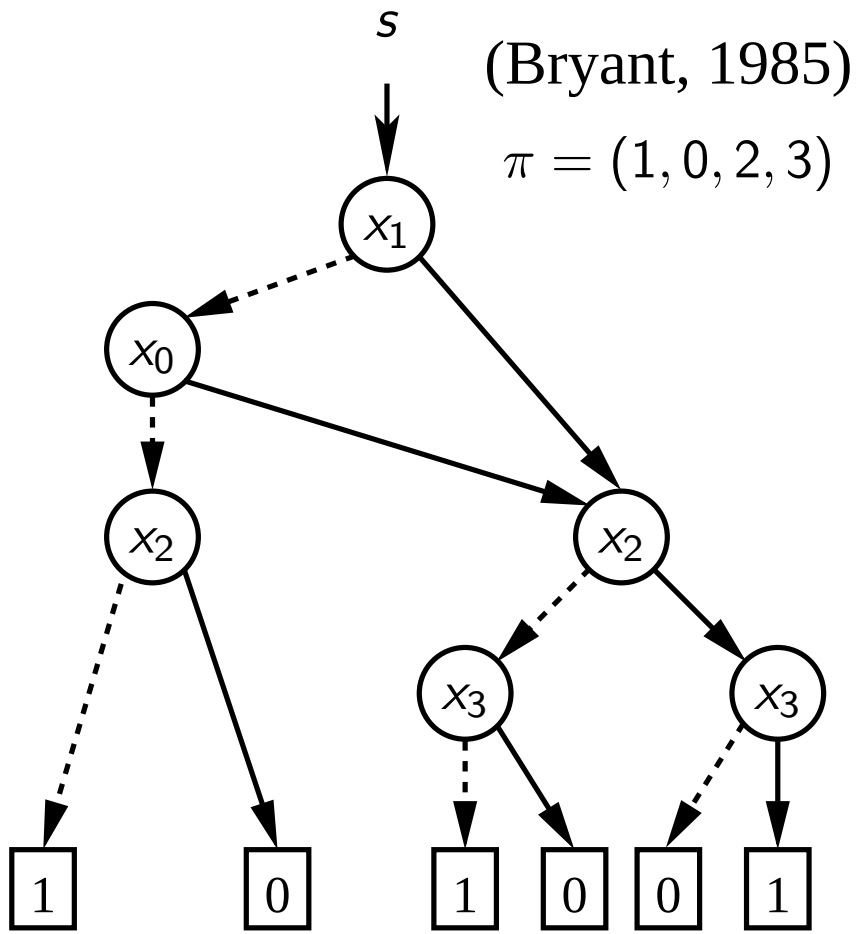

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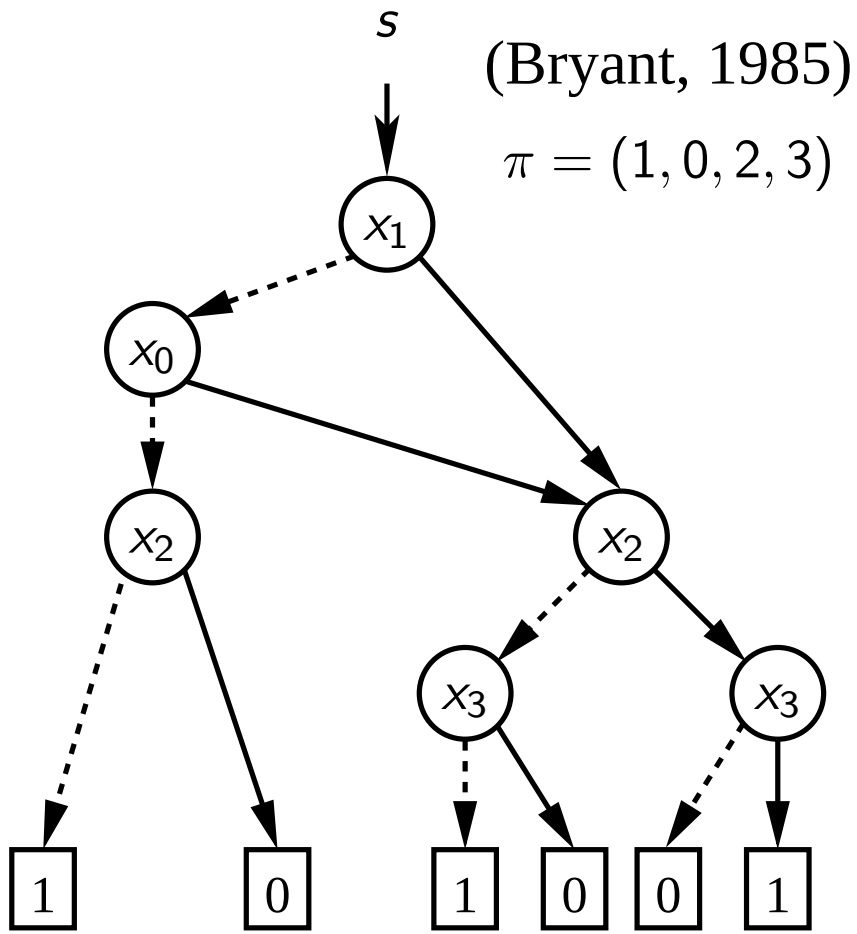

Algorithmic Properties of OBDDs

- Every function f on m vars. has at most OBDD size $\mathcal{O}(2^m/m)$.
- Hope for structured functions: OBDD size poly(m)
- Efficient operations for OBDDs $\mathcal{G}_f$ and $\mathcal{G}_h$:
  - Satisfiability: $f \not\equiv 0$
  - Equivalence: $f = h$
  - Variable replacement: $f_{|x_i=0/1}$
  - Binary synthesis: $f \otimes h$ for $\otimes = \vee, \wedge, \oplus, \ldots$
  - Quantification: $(\exists/\forall x_i)f$

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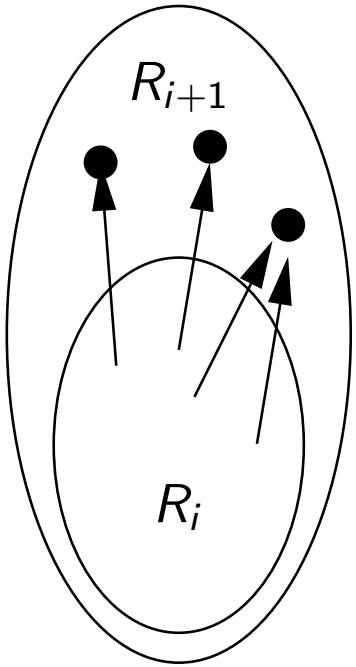
Implicit Graph Algorithms: An Example

Example: An implicit BFS algorithm on χ_G for

$$\chi_G(x, y) = 1 \Leftrightarrow (v_{|x|}, v_{|y|}) \in E$$

$i := 0$; $R_0(x) := (|x| = s)$
repeat
  $N(x) := (\exists y)[\chi_G(y, x) \wedge R_i(y) \wedge \overline{R_i(x)}]$
  $R_{i+1}(x) := R_i(x) \vee N(x)$
  $i := i + 1$
until $R_i = R_{i-1}$



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  - OBDDs well established in CAD, Model Checking, ...
  - Pure heuristics for mostly application-specific problems
  - No theoretical analyses of time/space
- Recent contributions:
  - Implicit algorithms for many graph-theoretic problems (Flows, Shortest Paths, Topological Sorting, ...)
  - Polylogarithmic upper bounds for structured instances
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- This talk:
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  - Fixed-parameter intractability of basic graph problems

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- Many works just consider the number of operations (SCCs, Gentilini et al., SODA'03).
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- New result: Impossible for P-complete problem (unless P=NC)!

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A Framework for Implicit Algorithms

Definition

A symbolic register access machine (SRAM) is a RAM with additional symbolic regs. $S_0, S_1, \ldots$ each holding a Boolean function $f : \{0,1\}^m \to \{0,1\}$. It offers ops. to

- get/increase m,
- evaluate S_i due to $a \in \{0,1\}^m$,
- read f from standard registers into S_i.
- copy/negate symbolic registers,
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each of cost 1. Input χ_I and output χ_O are located in S_0.

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- read f from standard registers into S_i.
- copy/negate symbolic registers,
- compute $S_i \otimes S_j$,
- $[\ldots]$

each of cost 1. Input χ_I and output χ_O are located in S_0.

A Framework for Implicit Algorithms

- SRAM model captures capabilties of "all" implicit (OBDD-based) algorithms.
- Implicit algorithm with $t(n)$ operations $\Rightarrow$ SRAM with time $\mathcal{O}(t(n))$.

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SRAM on input χ_I with time $t(n)$ and $m \leq k \log n$ variables can be simulated by PRAM in parallel time $\mathcal{O}((t(n))^2 \cdot \log^2 n)$ with $\mathcal{O}(n^k)$ processors on $I \in \{0,1\}^n$.

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Sketch of proof:

- Handle values $S_0(a), \ldots, S_r(a)$ by processor P_a for $a \in \{0,1\}^m$ and $r \leq t(n)$.
- $\Rightarrow \mathcal{O}(2^m) = \mathcal{O}(n^k)$ processors.
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Result for P-complete Problems

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P-complete problems have no PRAMs with time $\mathcal{O}(\log^k n)$ on $\mathcal{O}(n^k)$ processors unless $P = NC$.

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P-complete problems have no implicit algorithms with $\mathcal{O}(\log^k n)$ functional operations on $\leq k \log n$ variables unless $P = NC$.

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- Open: Is 0-1 flow maximization P-complete?
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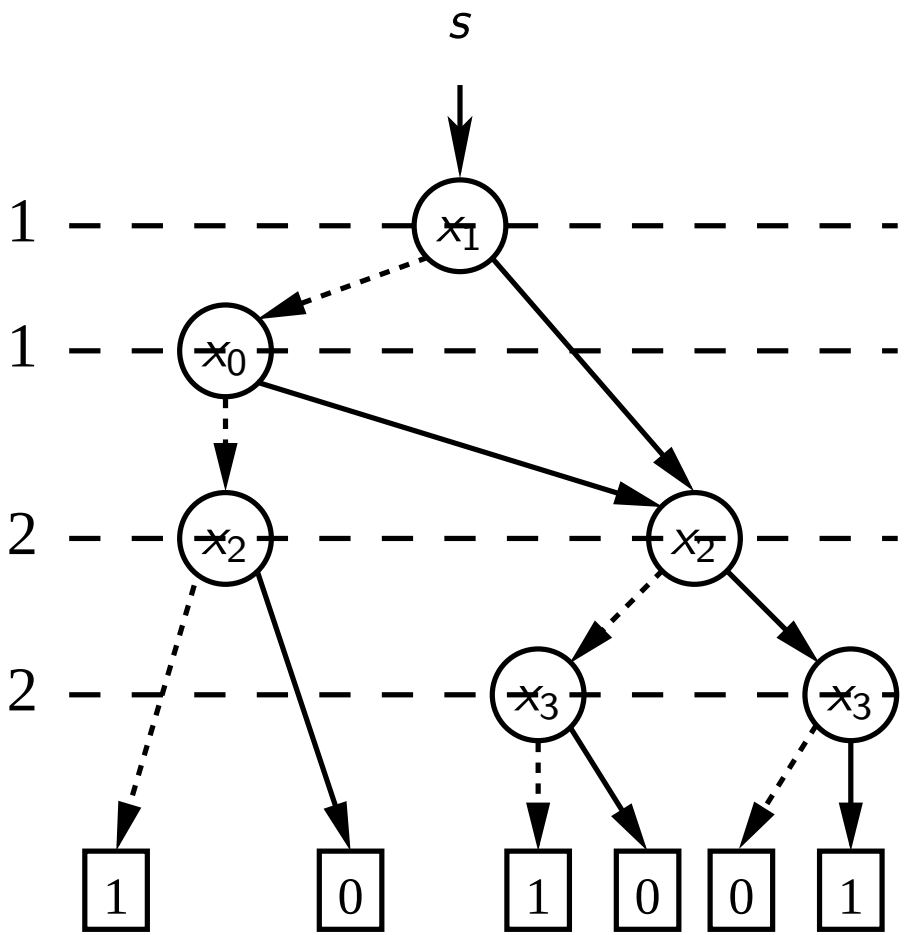
Definition of OBDD Width

Definition

The OBDD width is the maximum number of nodes labeled the same variable.



OBDD Width as Fixed Parameter

- Are there efficient algorithms for inputs with small OBDD width W?
- For width W of χ_G and some function α:
- Parameterized complexity $\mathcal{O}(\log^k |V| \cdot \alpha(W))$ possible?
- Feigenbaum proof: $W = \mathcal{O}(1) \Rightarrow$ No FPT-algo. for s-t-conn.
- New contribution: Fixed-parameter intractability for further problems on OBDD-represented graphs.

Theorem

None of the problems s-t-conn., APSP, MaxFlow, Acyclicity, Connectivity, Bipartiteness, Eulerian-Cycle, and Planarity on OBDD-represented graphs has an FPT-algo. unless P=PSPACE.

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Width-Preserving Reductions from Π to Π'

Map ϕ is width-preserving reduction from Π to Π' iff

- it maps OBDD χ_G to OBDD $\chi_{G'}$ with

$$G \in \Pi \Leftrightarrow G' \in \Pi',$$

- width W' of $\chi_{G'}$ depends only on W rather than on $|V|$.

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A sequence of $\mathcal{O}(1)$ arbitrary OBDD operations is width-preserving.

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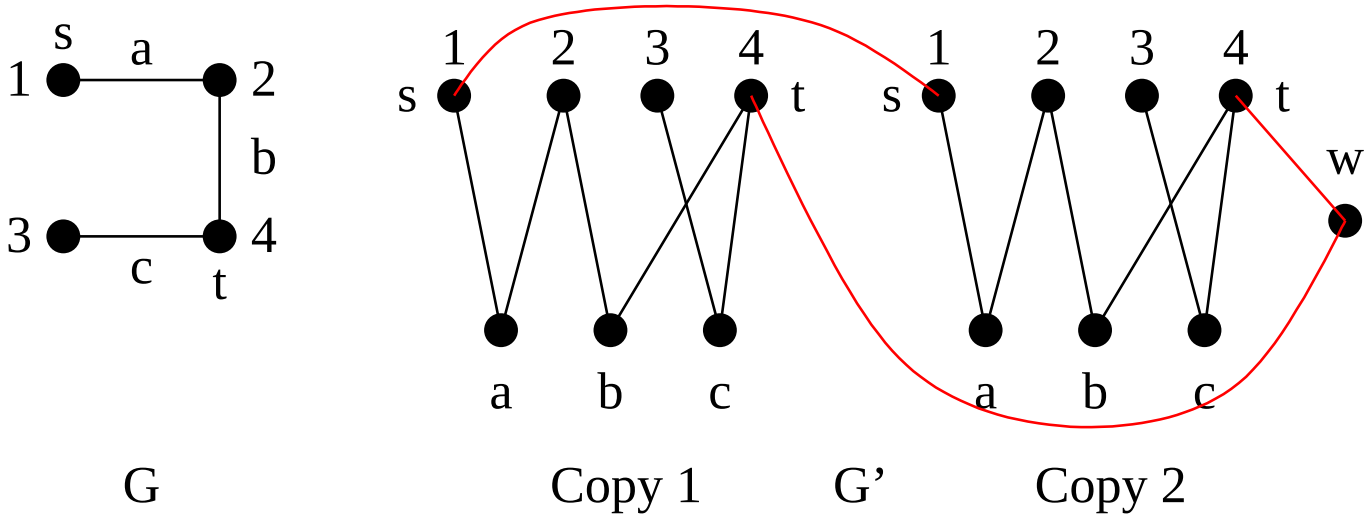
Fixed-Parameter Intractability of Bipartiteness

Exemplarily reduction from undir. s-t-conn. to bipart.:



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$$\begin{aligned}\chi_{G'}(x,y) := &\big[(T(x)=v)\wedge(T(y)=e)\wedge(i(x)=i(y))\wedge(c(x)=c(y))\wedge\chi_G(i(y),j(y))\big]\\ \vee &\big[(T(x)=e)\wedge(T(y)=v)\wedge(j(x)=j(y))\wedge(c(x)=c(y))\wedge\chi_G(i(x),j(x))\big]\\ \vee &\big[(T(x)=T(y)=v)\wedge(v_{|i(x)|}=v_{|i(y)|}=s)\wedge(c(x)\neq c(y))\big]\\ \vee &\big[(T(x)=v)\wedge(T(y)=w)\wedge(v_{|i(x)|}=t)\big]\ ,\end{aligned}$$

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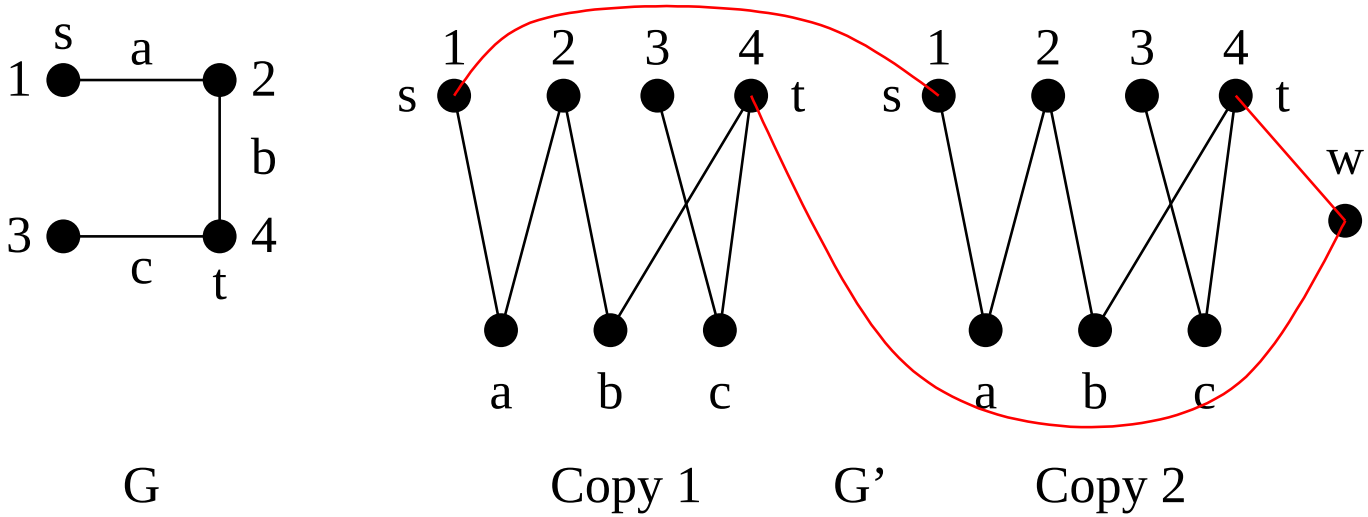

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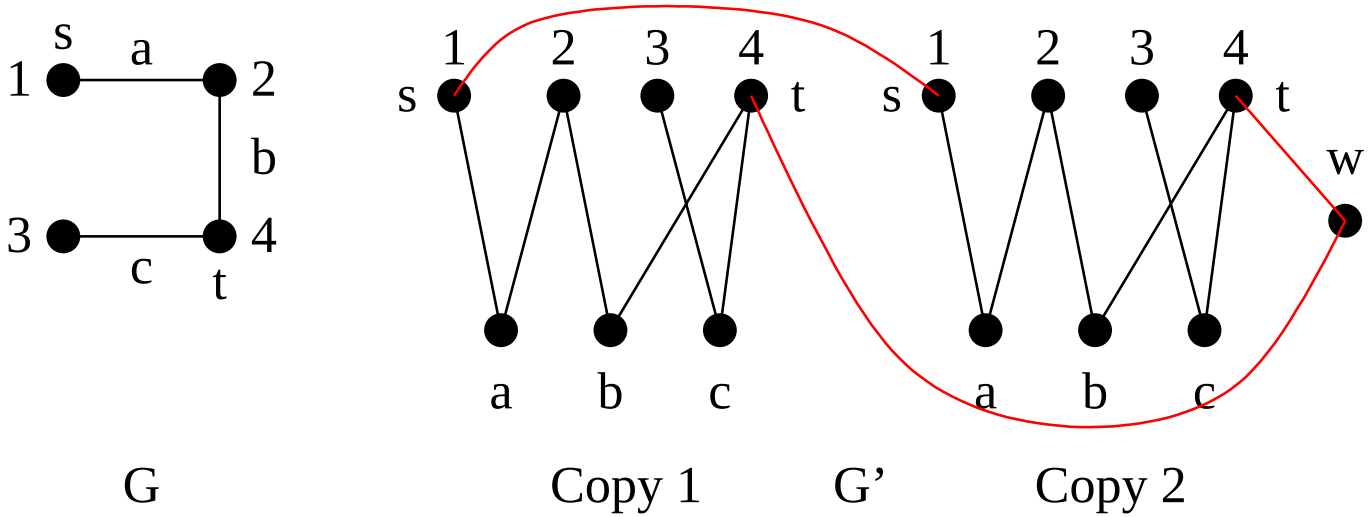

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