Small independent edge dominating sets in graphs of maximum degree three

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Problem definition

- Given a connected undirected graph $G = (V, E)$
- Find a minimum set of edges $E^{'} \subseteq E(G)$ which fulfils the following conditions:
  - for every edge $(u, w) \in E(G)$ either $(u, w) \in E^{'}$ or one of $(x, u), (w, y) \in E(G)$ belongs to $E^{'}$
  - no two edges of $E^{'}$ share a common endpoint.

Arbitrary graphs – results

- Minimum independent edge dominating set problem is NP-complete even when restricted to planar or bipartite graphs of maximum degree three
  M. Yannakakis, F. Gavril, Journal on Appl. Math. 1980

- The problem remains NP-complete for planar bipartite graphs, line graphs, total graphs, planar cubic graphs
  J. D. Horton, K. Kilakos, Journal on Disc. Math. 1993

- There is a polynomial time algorithm for trees
  M. Yannakakis, F. Gavril, Journal on Appl. Math. 1980

Arbitrary graphs – approximation

- It is NP-hard to approximate the problem within any factor smaller than 7/6 in general graphs and smaller than 1+1/487 in cubic graphs
  M. Chlebík, J. Chlebíková, Algorithms and Computations 2003

- Constructing any maximal matching gives approximation ratio of 2 for the problem

Cubic graphs – results

- The minimum independent edge dominating set of an n-vertex cubic graph is at most 9n/20+O(1)
  W. Duckworth, N. C. Wormald, 2000

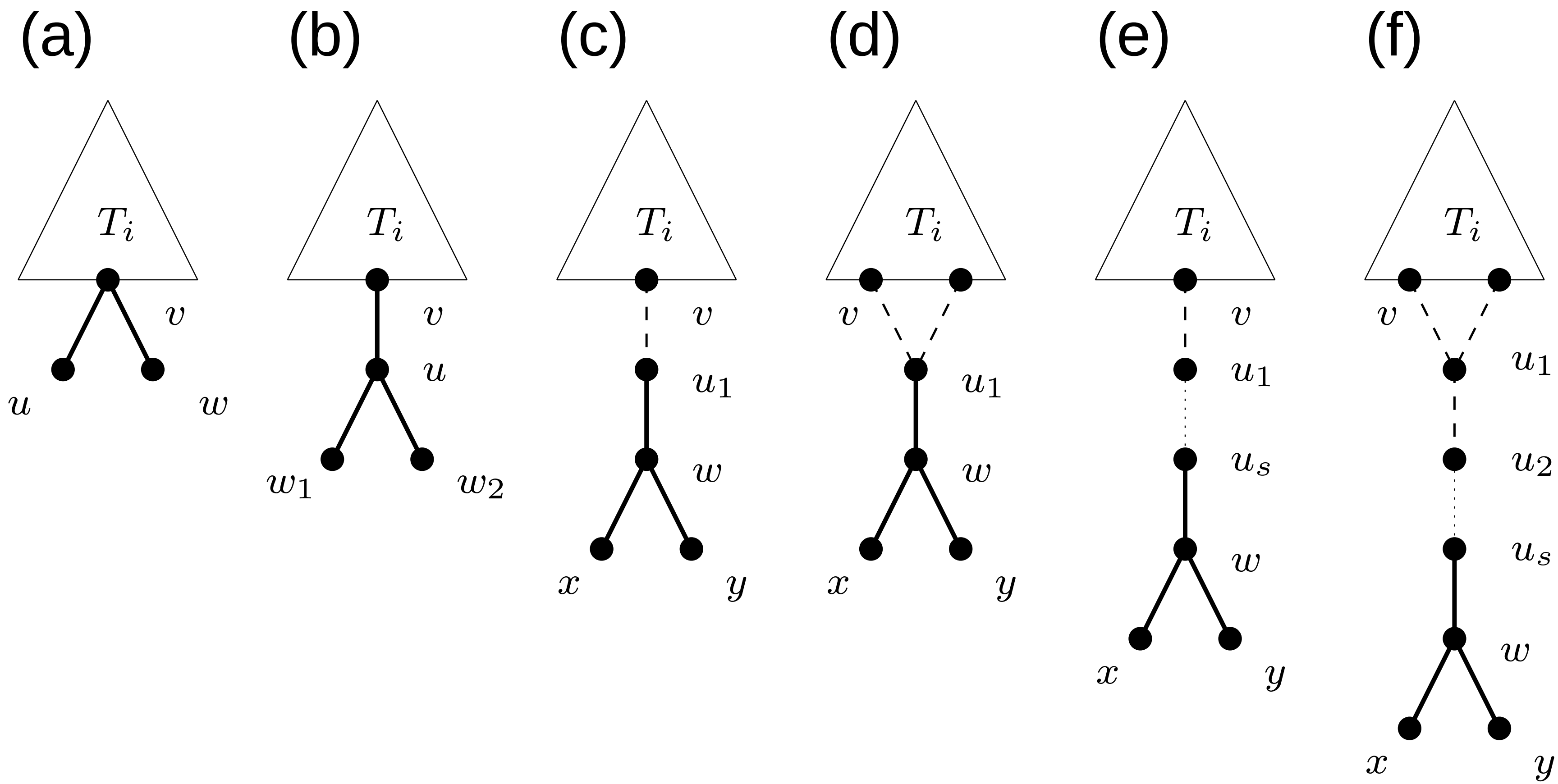
- There are families of cubic graphs for which the size of the minimum independent edge dominating set is at least 3n/8
  W. Duckworth, N. C. Wormald, 2000

- The lower bound for the size of the minimum independent edge dominating set is 3n/10

Our results

- An approximation algorithm which finds the independent edge dominating set of size at most 4n/9+1/3 in an n-vertex graph of maximum degree three. As a collorary we obtain an approximation ratio of 40/27 for the problem in cubic graphs.

Rules



Construction of the forest F

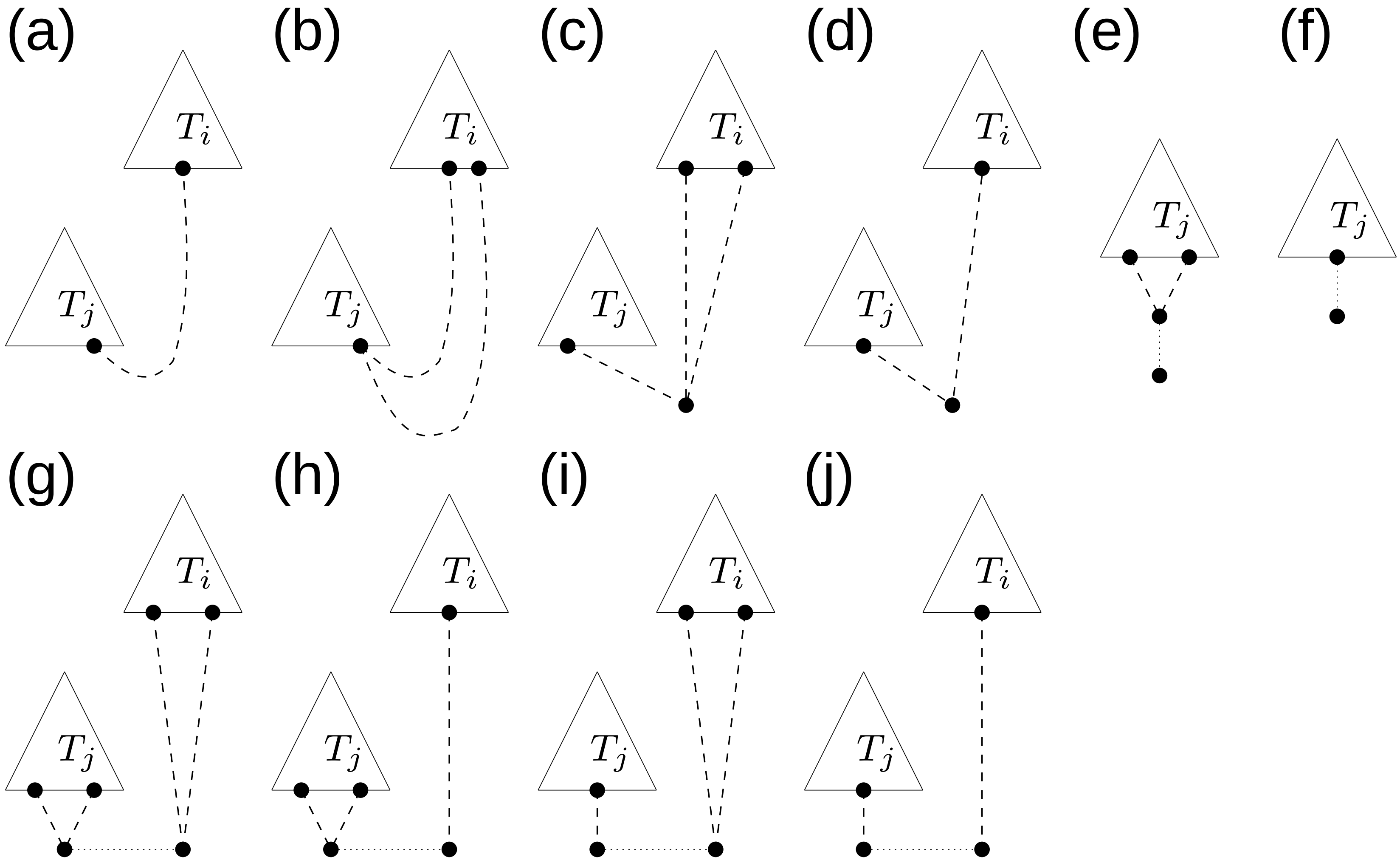
1. $F \leftarrow \emptyset$
2. $V(T_0) \leftarrow \{r_0, v_1, v_2, v_3\}$, where r_0 is any degree three vertex of G and $v_1, v_2, v_3 \in \Gamma_G(r_0)$; $E(T_0) \leftarrow \{(r_0, v_1), (r_0, v_2), (r_0, v_3)\}$; let r_0 be a root of T_0; $i \leftarrow 0$
3. **if it is possible** find the leftmost leaf in T_i that can be expanded by the Rule 1 and expand it; go to the step 3;
   else go to the step 4;
4. **if it is possible** find the leftmost leaf in T_i that can be expanded by the Rule 2 and expand it; go to the step 3
   else $F \leftarrow F \cup T_i$ and go to the step 5
5. **if it is possible** find the leftmost leaf v in T_i such that Rule 3 can be applied to v and apply this rule to v; $i \leftarrow i + 1$; let T_i be a new tree created in this step; go to the step 3 with T_i
   else go to the step 5 with the father of T_i

Properties of F

Fact 1 *Let* $v \in V(T_i), T_i \in F$.

1. *If* $deg_{T_i}(v) = 2$ *and* $w \in \Gamma_{T_i}(v)$ *then* $deg_{T_i}(w) = 3$.
2. *If* $deg_{T_i}(v) = 2$ *and* $w \in \Gamma_G(v)$ *then* $w \in V(T_i)$.
3. *If* v *is adjacent to the vertex* $w \in EX \cup R$, $u \in \Gamma_G(v)$ *and* $u \neq w$ *then* $u \in V(T_i)$.

The edges $e \in E(G) \setminus E(F)$



Notation

- $G^{'}$ – current graph
- $F^{'}$ – current forest
- $EX^{'} = V(G^{'}) \backslash V(F^{'})$
- $S^{'}_{x}$ – the subtree of $T^{'} \in F^{'}$ rooted at some vertex $x \in V(T^{'})$
- $EDS(G)$ – independent edge dominating set contructed by our algorithm

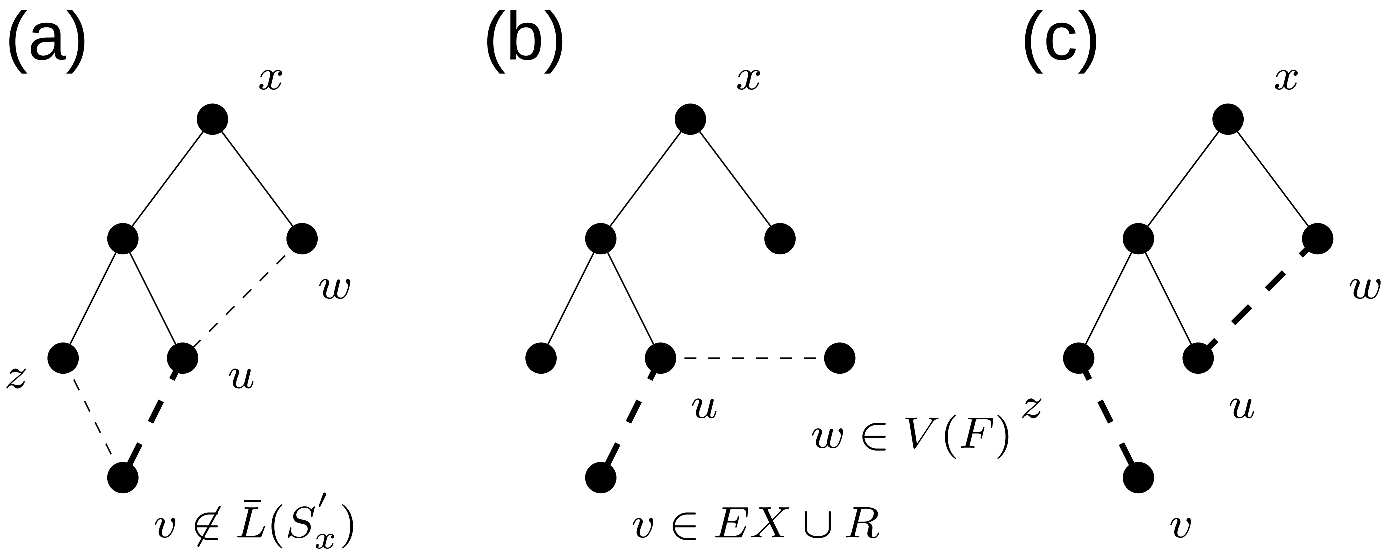
$DOMINATE_i$

- DOMINATE$_1(S_x^{'} \cup S_y^{'})$ dominates the edges of the subtrees $S_x^{'}$, $S_y^{'}$, where $deg_T(x) = 2$ and x is the endpoint of the edge $(x,\, y) \in E(G) \backslash E(F)$ together with edges $e \in E(G^{'}) \backslash E(F^{'})$ incident to $V(S_x^{'}) \cup V(S_y^{'})$.
- DOMINATE$_2(S_x^{'})$ which dominates the edges of some paths which end at the leaves of $S_x^{'}$ and go through the edges $e \in E(G^{'}) \backslash E(F^{'})$.
- DOMINATE$_3(S_x^{'})$ dominates $E(S_x^{'})$ and all the edges incident to $V(S_x^{'})$.

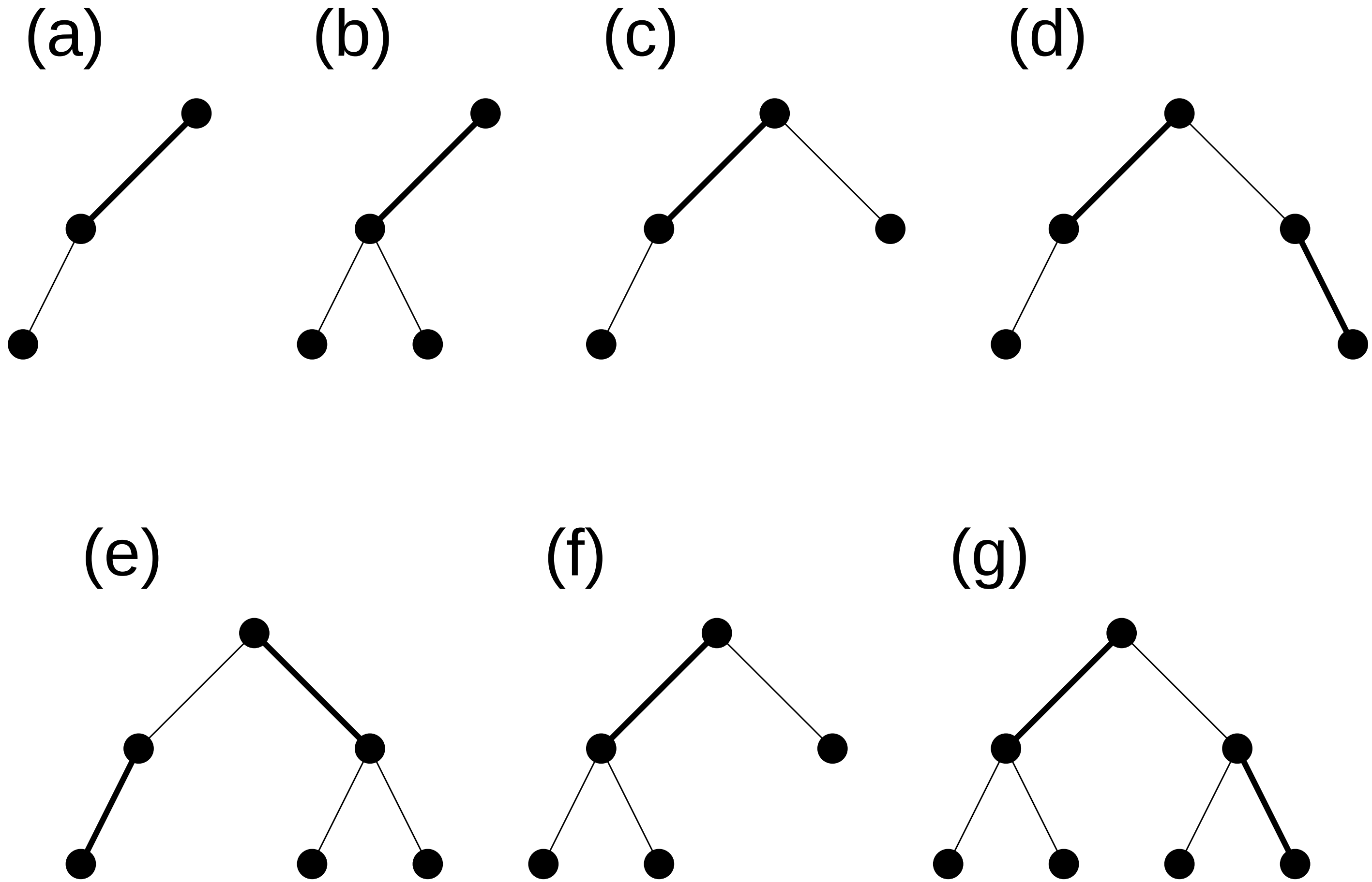
Main property

Property 1 *The procedure* DOMINATE$_i()$, $i \in \{1,\, 2,\, 3\}$ *adds* $\bar{m}$ *edges to* $EDS(G)$ *and removes* $\bar{n}$ *vertices from* $V(G^{'})$, *where* $\bar{m} \leq \frac{4}{9}\bar{n}$. *The only exception may be the case when* DOMINATE$_3()$ *is applied to the tree rooted at* r_0 *or at the son of* r_0; *then* $\bar{m} \leq \frac{4}{9}\bar{n} + \frac{1}{3}$.

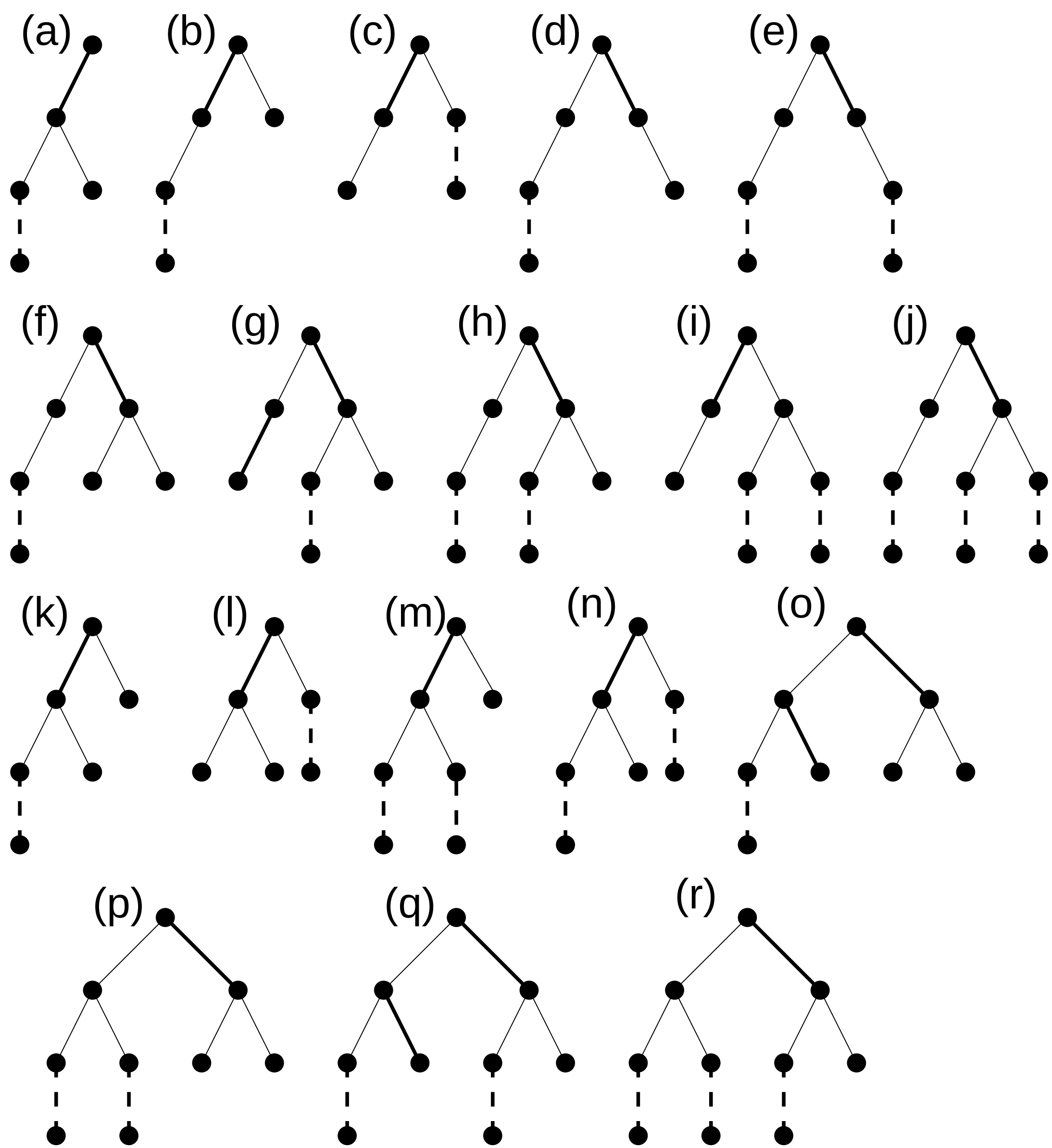
Required edges



$DOMINATE_3(S'_x)$ - *basic*



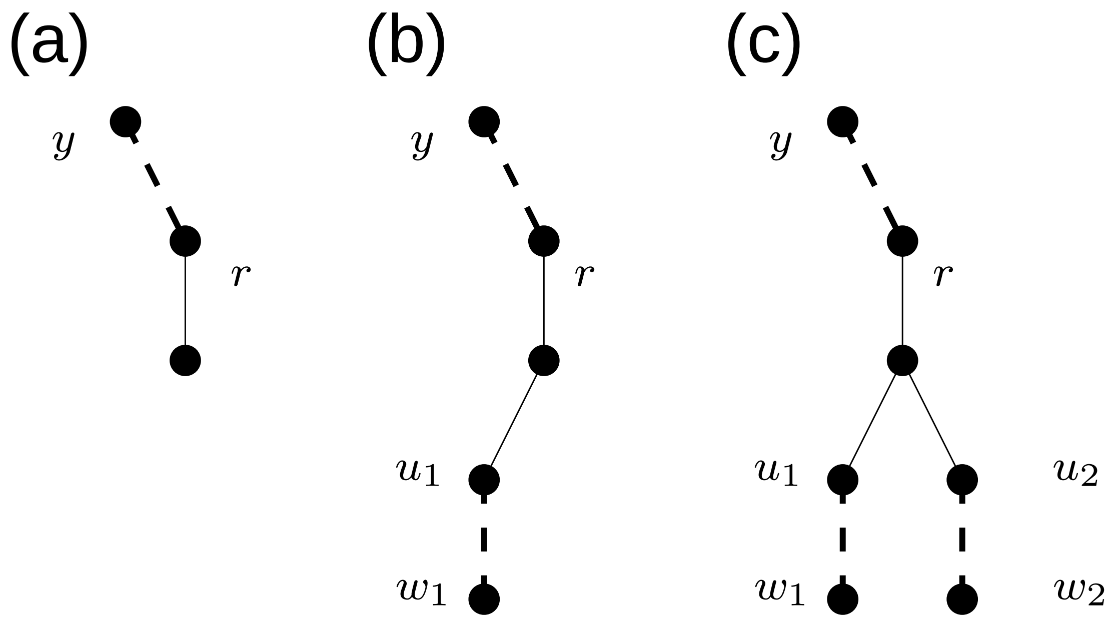
$\mathit{DOMINATE}_3(S'_x)$ *when* $h(S'_x) = 2$



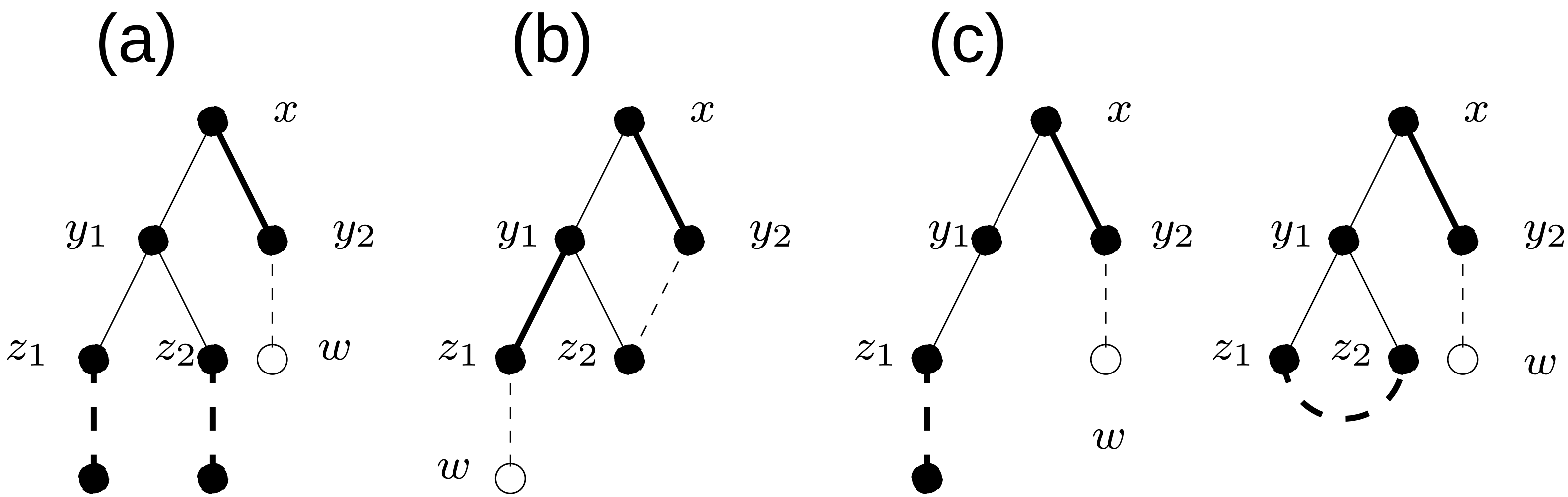
$S_x^{''}$ definition

Let $S_x^{''}$ be a subgraph of $S_x^{'}$ which fulfils the following condition: if two edges $(u_1, u_2), (v_1, v_2) \in E(S_x^{'})$ where either the path from u_2 to x in $T^{'}$ is shorter than the path from v_2 to x in $T^{'}$ or u_2 is on the left of v_2 in $S_x^{'}$, are incident to an edge $(u_2, v_2) \in REQUIRED(S_x^{'})$, then the edge $(u_1, u_2) \notin E(S_x^{''})$.

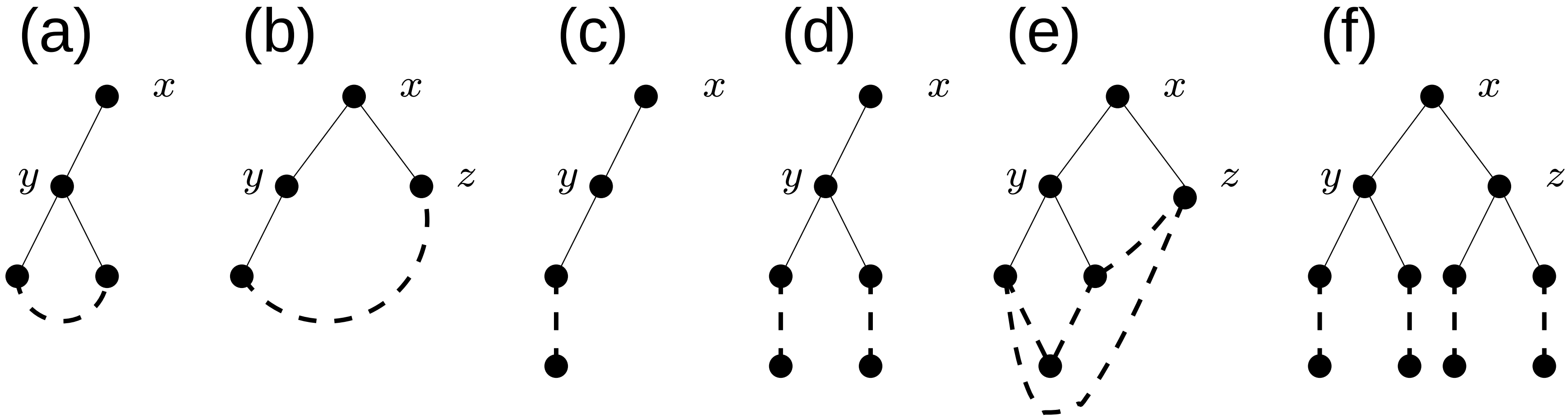
$DOMINATE_3(S'_x)$ *when* $h(S'_x) = 2$ *and* $x = r$



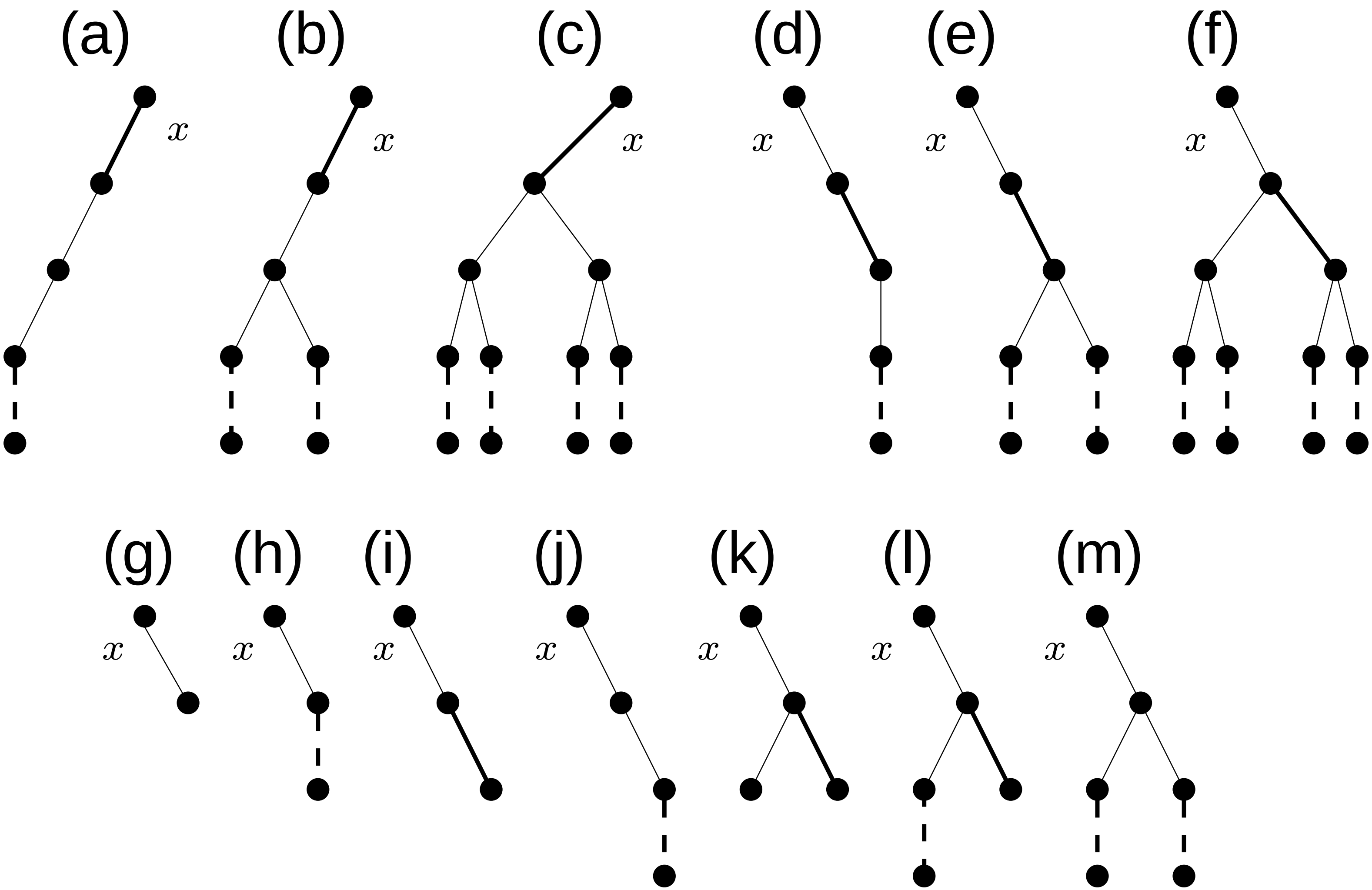
$DOMINATE_3(S_x^{'})$ *when* $h(S_x^{'}) = 2$



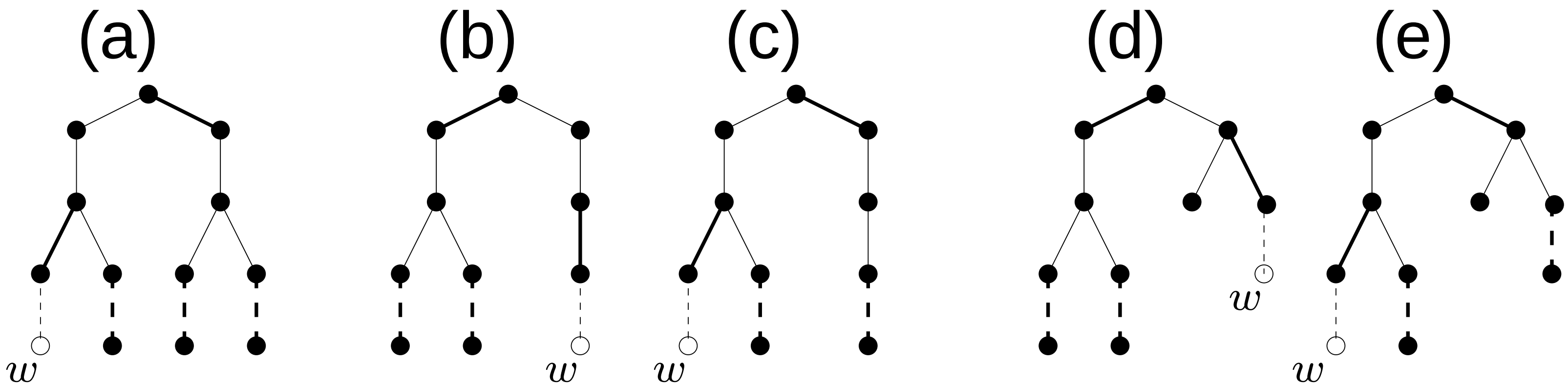
Impossible to apply DOMINATE$_3(S'_x)$



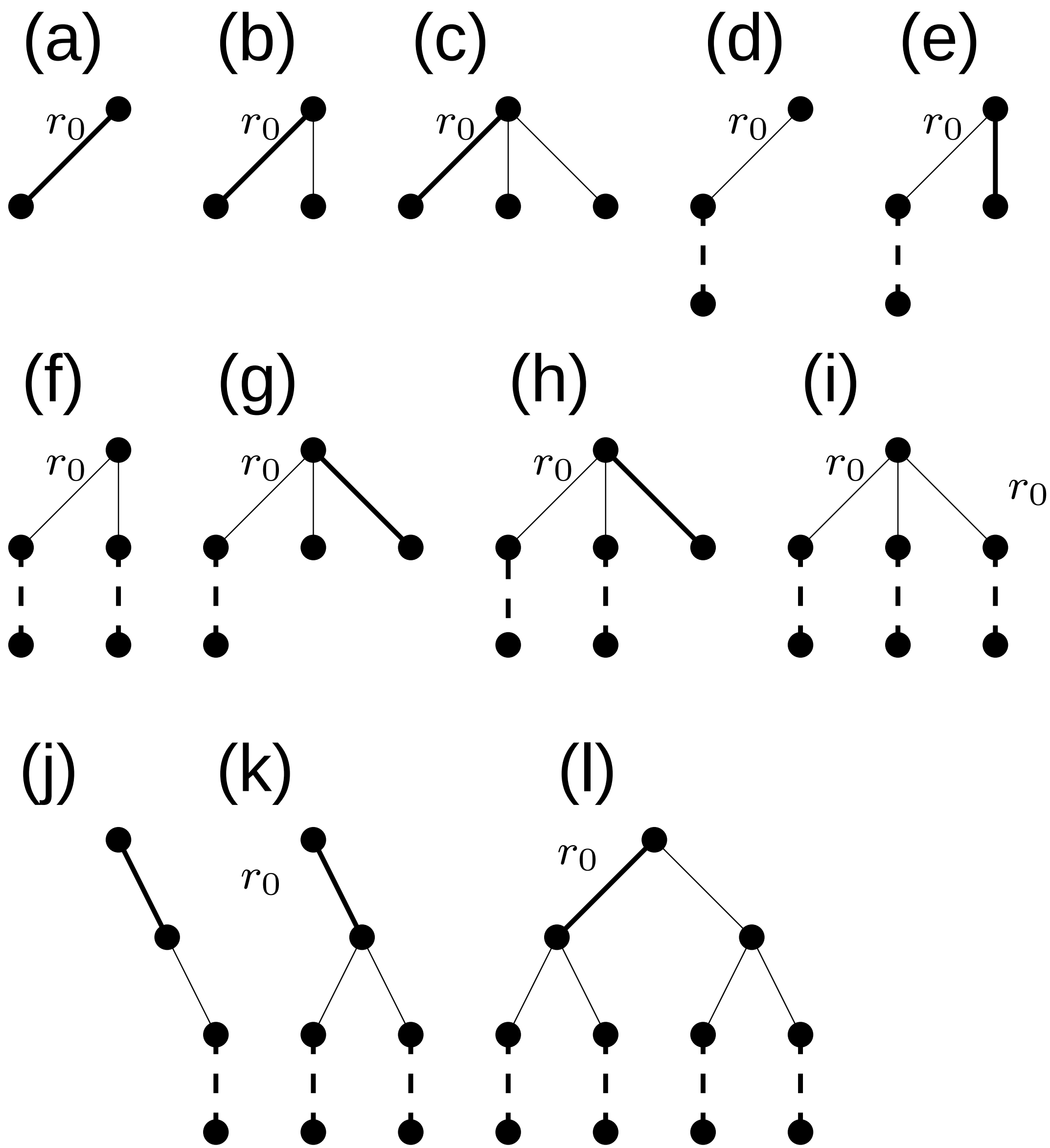
$DOMINATE_3(S'_x)$ *when* $h(S'_x) = 3$



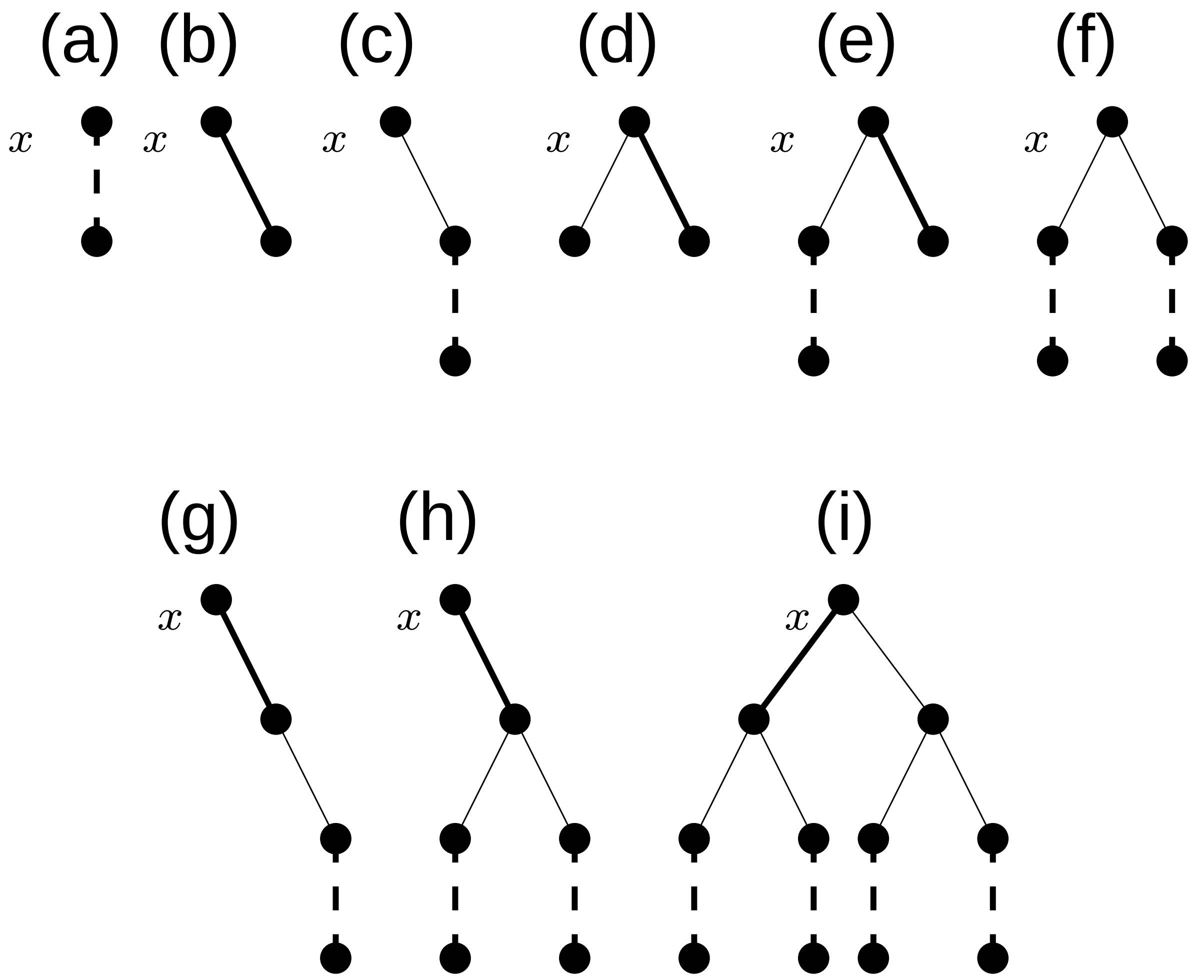
$DOMINATE_3(S_x^{'})$ *when* $h(S_x^{'}) = 3$



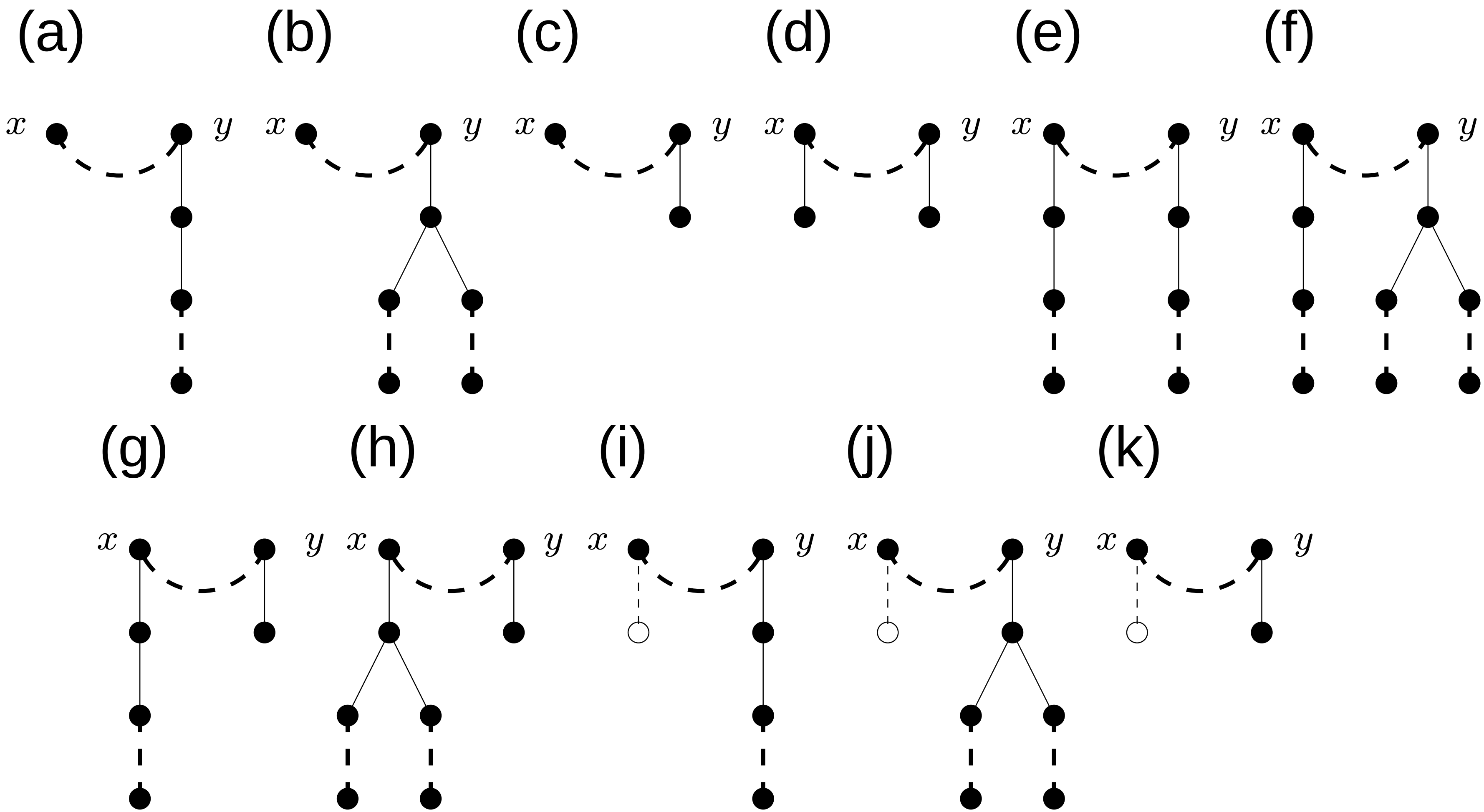
$DOMINATE(S'_x)$ *when* $x = r_0$



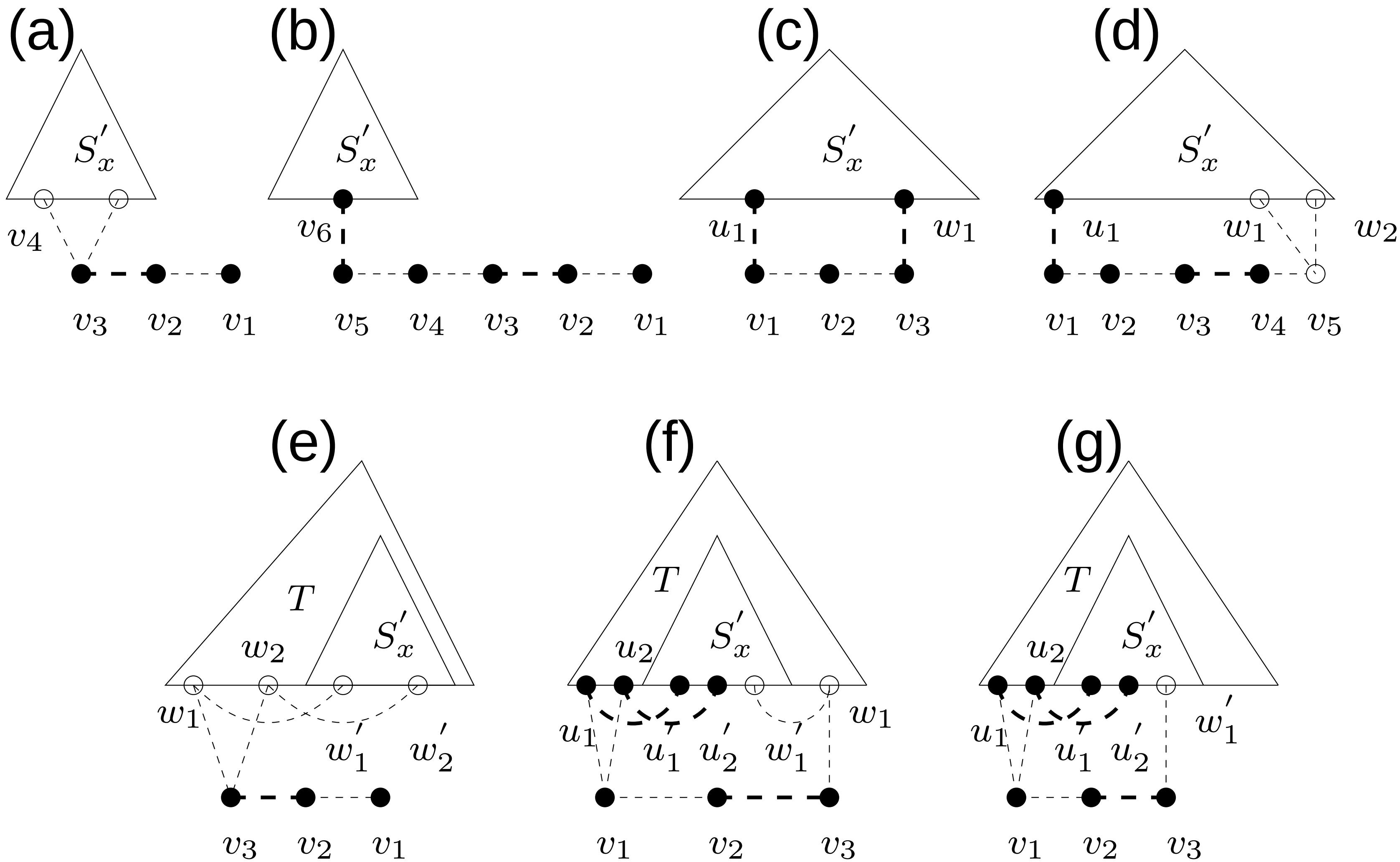
DOMINATE(S'_x) *when* x *is the last son of* r_0



$DOMINATE_1(S'_x \cup S'_y)$



$DOMINATE_2(S'_x)$



EDS_T($T^{'}$)

1. apply procedure EDS_T() to the sons $T_i^{'} \in F^{'}$ of $T^{'}$ (successively from the leftmost to the rightmost);
2. EDS($T^{'}$).

EDS($S_x^{'}$)

Let $W = \{u \in V(S_x^{'}) :\ deg_F(u) = 2$ and $deg_{G'}(u) = 3\}$.

1. while $W \neq \emptyset$
   (a) apply in-order search and find the first vertex $u \in W$;
   let w be such a vertex that $(u,\ w) \in E(G)\backslash E(F)$;
   (b) EDS($S_u^{'}$);
   (c) if $w \in V(F^{'})$: EDS($S_w^{'}$);
   (d) if $deg_{G'}(u) = 3$ or $deg_{G'}(w) = 3$: MARK_REQUIRED($S_u^{'} \cup S_w^{'}$);
   DOMINATE_PATHS($S_u^{'} \cup S_w^{'}$); DOMINATE$_1$ ($S_u^{'} \cup S_w^{'}$);
2. MARK($S_x^{'}$).

MARK_REQUIRED($S_x^{'}$)

Let $W = \{(u, w) \in E(G^{'}) \backslash E(F^{'}) : u \in \bar{L}(S_x^{'})\}$.

1. $REQUIRED(S_x^{'}) := \emptyset$
2. while there is a vertex $u \in \bar{L}(S_x^{'})$ and the edges $(u, v), (u, w) \in W$, $v \in \bar{L}(S_x^{'})$, $w \notin \bar{L}(S_x^{'})$:
   find the first u, add (u, w) to $REQUIRED(S_x^{'})$ and remove the edges incident to u, w from W.
3. while there is a vertex $u \in \bar{L}(S_x^{'})$ and the edge $(u, w) \in W$, $w \in EX \cup R$:
   find the first u, add (u, w) to $REQUIRED(S_x^{'})$ and remove the edges incident to u, w from W.
4. while there is a vertex $u \in \bar{L}(S_x^{'})$ and the edge $(u, w) \in W$:
   find the first u, add (u, w) to $REQUIRED(S_x^{'})$ and remove the edges incident to u, w from W.

DOMINATE_PATHS($S_x^{'}$)

1. if $TAILS_1(S_x^{'}) \neq \emptyset$ or $EARS_1(S_x^{'}) \neq \emptyset$: DOMINATE$_2$($S_x^{'}$);
2. MARK_REQUIRED($S_x^{'}$);
3. if $TAILS_2(S_x^{'}) \neq \emptyset$ or $EARS_2(S_x^{'}) \neq \emptyset$ or $EARS_3(S_x^{'}) \neq \emptyset$: DOMINATE$_2$($S_x^{'}$); goto 2.

MARK(S'_x)

1. if $h(S'_x) > 2$: apply procedure MARK() to the trees rooted at the sons of x (successively from the leftmost to the rightmost);
2. MARK_REQUIRED(S'_x);
3. DOMINATE_PATHS(S'_x);
4. if x has at most two sons u_i, $1 \leq i \leq 2$:
   (a) while DOMINATION_POSSIBLE(S'_{u_i}): DOMINATE$_3$(S'_{u_i});
   (b) if DOMINATION_POSSIBLE(S'_x): DOMINATE$_3$(S'_x); else: return;
5. if x has three sons u_i, $1 \leq i \leq 3$ (in this case $x = r_0$):
   (a) while DOMINATION_POSSIBLE(S'_{u_i}): DOMINATE$_3$(S'_{u_i});
   (b) if $h(S'_{r_0}) > 1$ and $deg_{G'}(r_0) = 3$: let $u_3 \notin V(\bar{S}'_{r_0})$;
      i. DOMINATE$_3$($\bar{S}'_{r_0}$);
      ii. if $u_3 \in V(G')$: MARK_REQUIRED(S'_{u_3}); DOMINATE$_3$(S'_{u_3});
   (c) else if $r_0 \in V(G')$: DOMINATE$_3$(S'_{r_0}).

Properties of F

Fact 2 *Let* $deg_T(u) = deg_T(w) = 2$*,* $T \in F$*,* $(u, w) \in E(G) \setminus E(F)$*,* $a =$ *LCA*$_T(u, w)$ *and let* u *be on the left of* w *in* T*. Then* w *is the right son of* a*, and there are no vertices of degree 2 on the path from* u *to* a *in* T*.*

Properties of F

Fact 3 *Let r be a root of a tree $T \in F$. Let $u \in \bar{L}(T)$ and let w be the first vertex of degree 2 on the path from u to r in T. If there is $v \in V(T)$ such that $deg_F(v) = 2$ and $(u, v) \in E(G) \backslash E(T)$ then w is an ancestor of v in T.*

Main theorem

Theorem 1 *Let G be a graph of maximum degree three, where $|V(G)| = n$. Our algorithm constructs for G an independent edge dominating set $EDS(G)$ of size at most 4n/9 +1/3 in linear time.*