Merging and Merge-sort in a Single Hop Radio Network

Marcin Kik

Wrocław University of Technology

Poland

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- n stations communicating by radio messages

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Energetic cost of the algorithm is the maximal energy dissipated by a single station.

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We assume that single key can be sent in a single message.

Sorting on single hop:

- From comparator networks: (each comparator simulated in two slots)
  - AKS energy: $\Theta(\lg n)$, time: $\Theta(n \lg n)$
  - Batcher energy: $\approx \lg^2 n$, time: $\Theta(n \lg^2 n)$

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- practical Merge-sort
  energy: $\frac{1}{2} \lg^2 n + \frac{7}{2} \lg n$,
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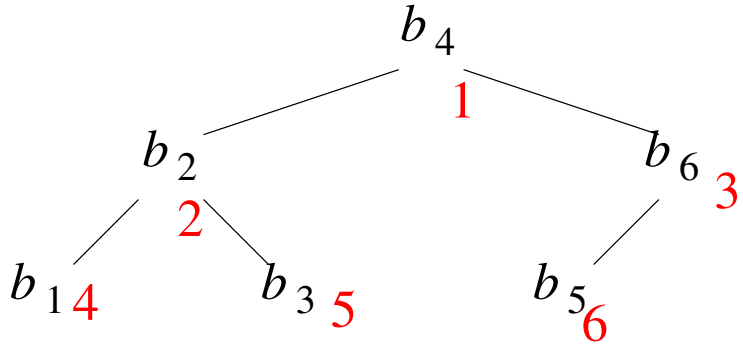
If we simulate the simple sequential algorithm, then the total energy for broadcasting and listening is $O(n)$, but it is not balanced.

Ranking

Ranking of $a_1 \ldots a_m$ in $b_1 \ldots b_m$:

- $a_1 \ldots a_m$ – sorted
- $b_1 \ldots b_m$ – sorted
  (balanced binary tree T_m)
- all values distinct
- in-order indexes
- "preorder" indexes – slots
- ranks
- timers – slots



a_1 a_2 a_3 a_4 a_5 a_6

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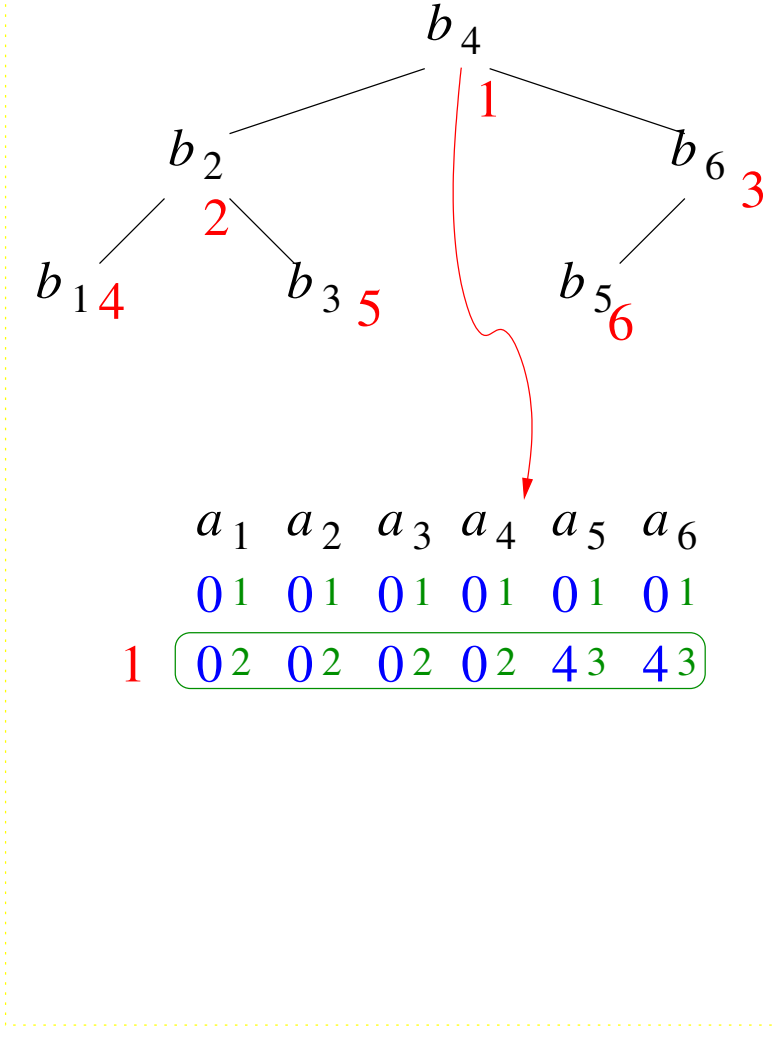

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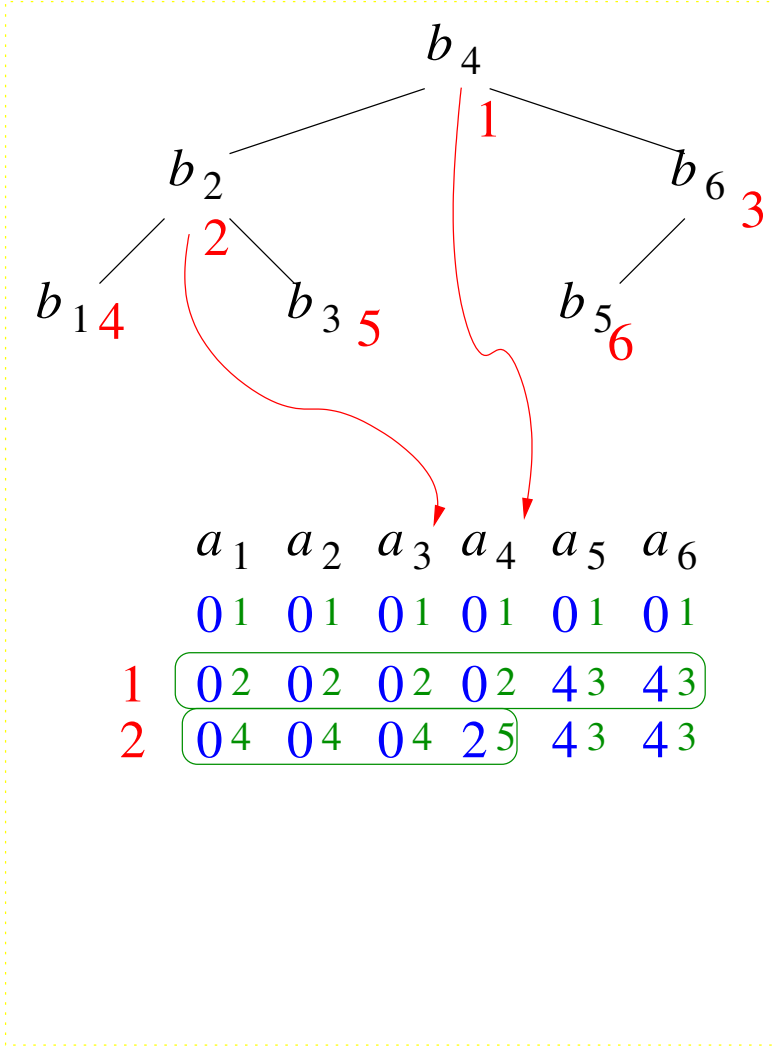

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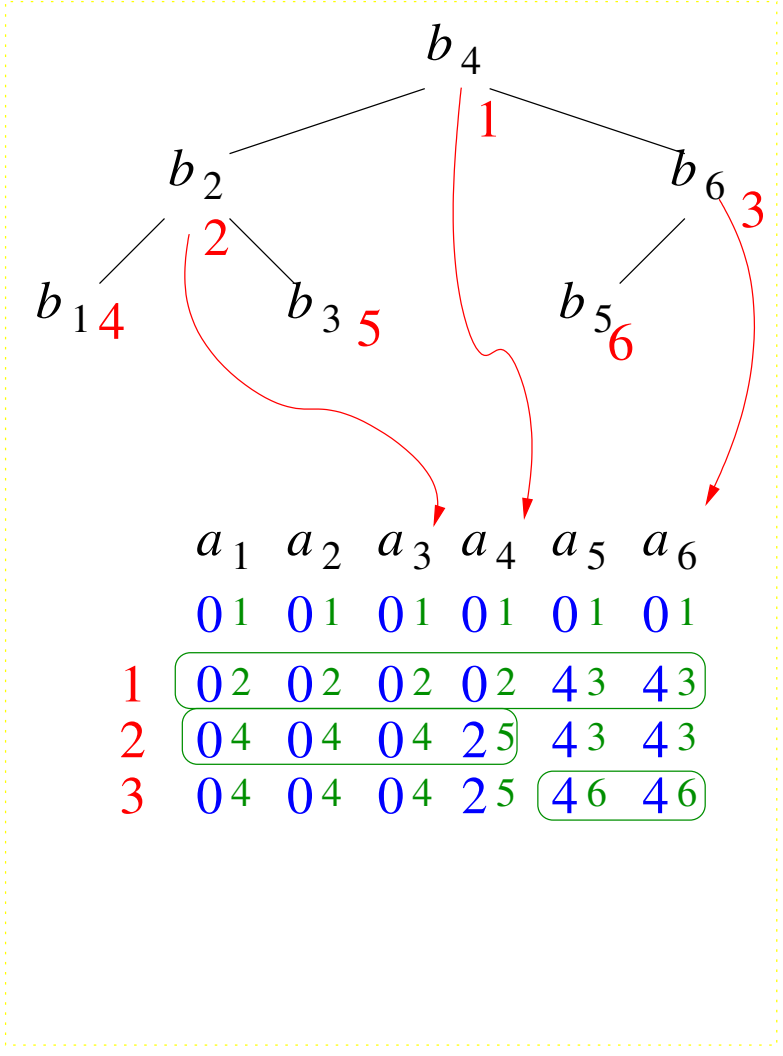

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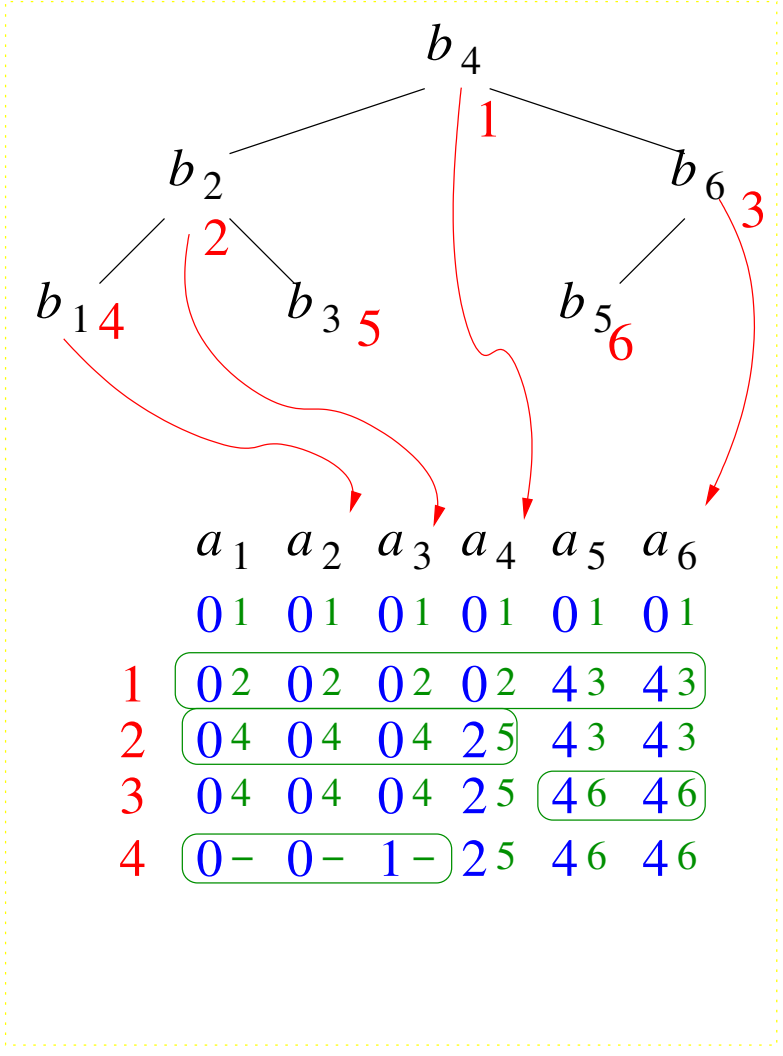

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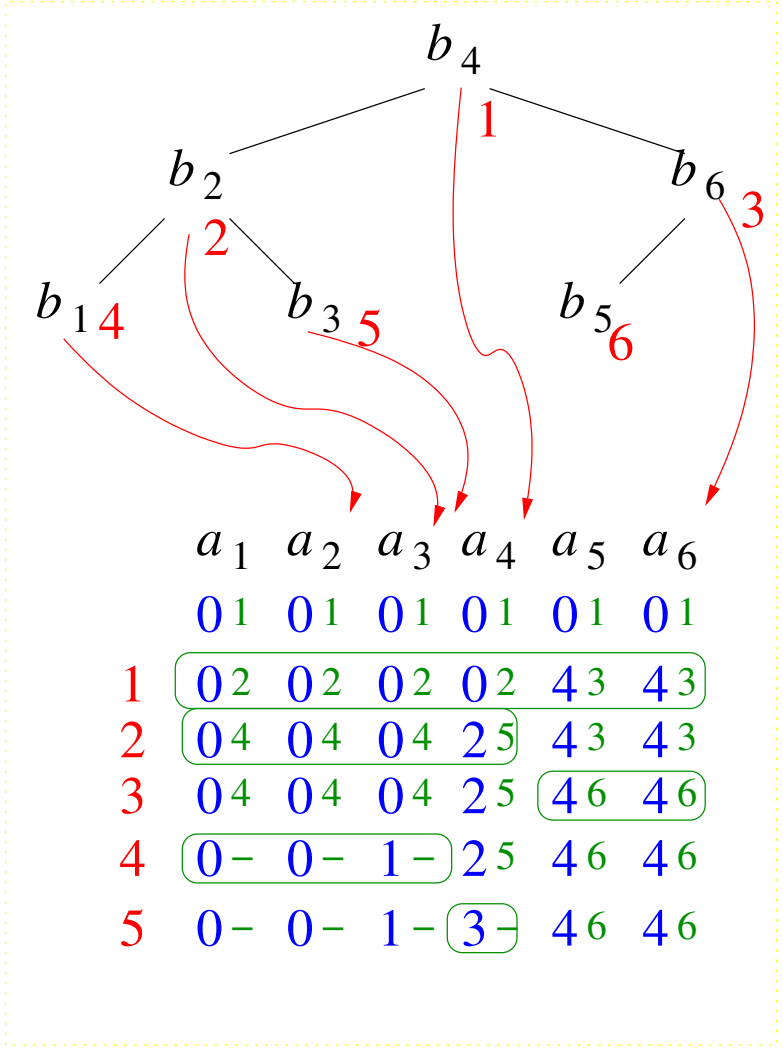

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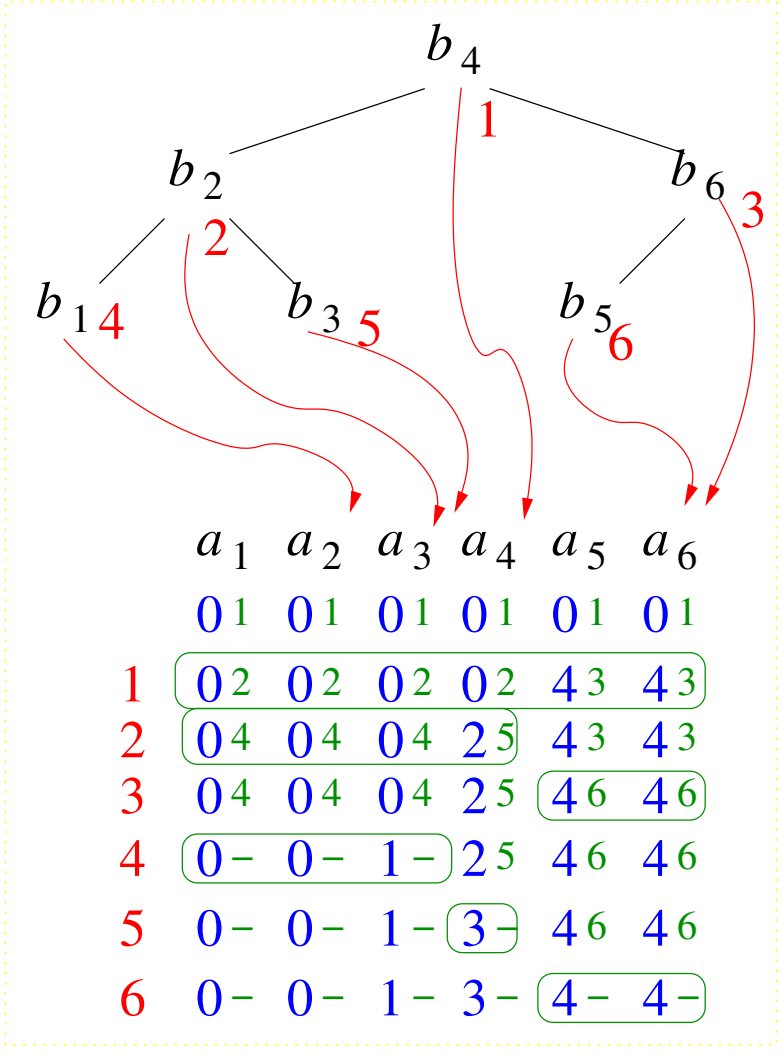

Ranking

procedure Rank($\langle a_1, \ldots, a_m \rangle$, $\langle b_1, \ldots, b_m \rangle$)
begin
  Each a_i does: $timer[a_i] \leftarrow 1$; $rank[a_i] \leftarrow 0$;
  for *time slot* $d \leftarrow 1$ **to** m **do**
    Let x be such that d is preorder index of x.
    b_x broadcasts its key k.
    Each a_j with $timer[a_j] = d$ listens and does:
    if $key[a_j] < k$ **then**
      $timer[a_j] \leftarrow$ preorder index of left child of x
    else
      $timer[a_j] \leftarrow$ preorder index of right child of x
      $rank[a_j] \leftarrow x$;
end

Energetic cost and time

Energy:

- Each b_i broadcasts once.
- Each a_i listens $\lceil \lg(m+1) \rceil$ (height of T_m)times

Time: m slots.

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- Each a_i and b_i computes final *index* of its *key* as: $i + ($*its rank in the other sequence*$)$.

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- Each a_i and b_i computes final *index* of its *key* as: $i + ($*its rank in the other sequence*$)$.
- `Permutation routing`: In step t, the element with *index*$= t$ broadcasts its *key* to the tth element of the merged sequence

Simple Merge

procedure Merge($\langle a_1, \ldots, a_m \rangle$, $\langle b_1, \ldots, b_m \rangle$)
begin
  Rank($\langle a_1, \ldots, a_m \rangle$,$\langle b_1, \ldots, b_m \rangle$);
  Rank($\langle b_1, \ldots, b_m \rangle$,$\langle a_1, \ldots, a_m \rangle$);
  Each a_i does: $idx[a_i] \leftarrow i + rank[a_i]$;
  Each b_i does: $idx[b_i] \leftarrow i + rank[b_i]$;
  For $1 \leq i \leq m$ let $c_i = a_i$ and $c_{m+i} = b_i$.
  for *time slot* $t \leftarrow 1$ **to** $2m$ **do**
    c_i with $idx[c_i] = t$ broadcasts its key k.
    c_t listens and does: $new[c_t] \leftarrow k$;
  Each c_i does: $key[c_i] \leftarrow new[c_i]$;
end

Merge: Energetic cost

Energy:

- First Rank:
  - each a_i: $\lceil \lg(m+1) \rceil$
  - each b_i: 1
- Second Rank:
  - each b_i: $\lceil \lg(m+1) \rceil$
  - each a_i: 1
- last for loop:
  - each station: 2 (once listens, once broadcasts)
- Totally: energ. cost: $\lceil \lg(m+1) \rceil + 3$

Merge: time

- First Rank: m
- Second Rank: m
- Last for loop: $2m$
- Total time: $4m$

Merge-sort

Obvious application of Merge: (Let $n = 2^k$)

- Energy

$$\begin{aligned} E(n) &= \sum_{l=0}^{k-1} (\lceil \lg(2^l + 1) \rceil + 3) \\ &= \frac{1}{2} \lg^2 n + \frac{7}{2} \lg n \end{aligned}$$

- Time

$$\begin{aligned} T(n) &= \sum_{l=0}^{k-1} n/(2 \cdot 2^l) \cdot (4 \cdot 2^l) \\ &= 2n \lg n \end{aligned}$$

Remark:

In all mergings except the last one:

Instead of the last “for” loop (i.e. `permutation routing`): Each station c_i internally temporarily modifies its global number to the number of the destination of its key and acts as its destination.

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- Energy

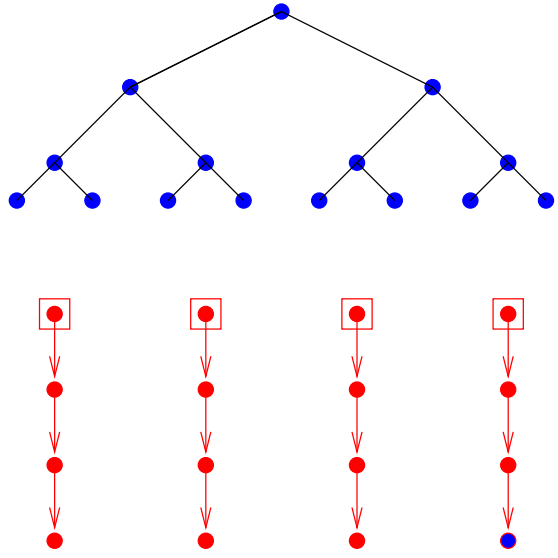
$$E(n) = \frac{1}{2}\lg^2 n + \frac{3}{2}\lg n + 2$$

- Time

$$T(n) = n\lg n + n$$

Balancing of listening energy

Each *block* of $\lceil \lg(m+1) \rceil$ *listeners* computes the rank of its *leader*. (The message to the next *listener*: *(leader's key, timer, rank)*)



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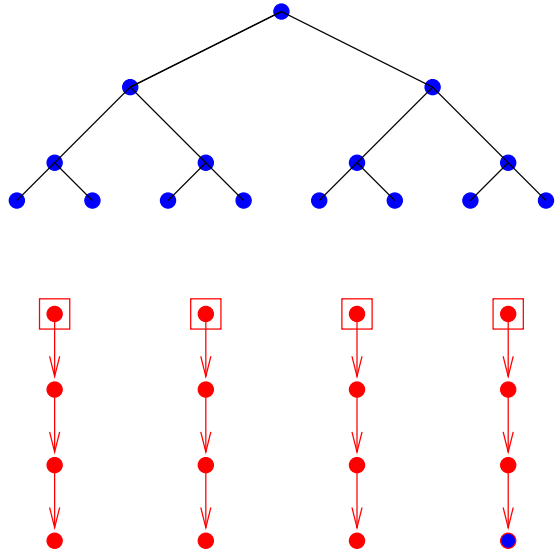
- listening: 2



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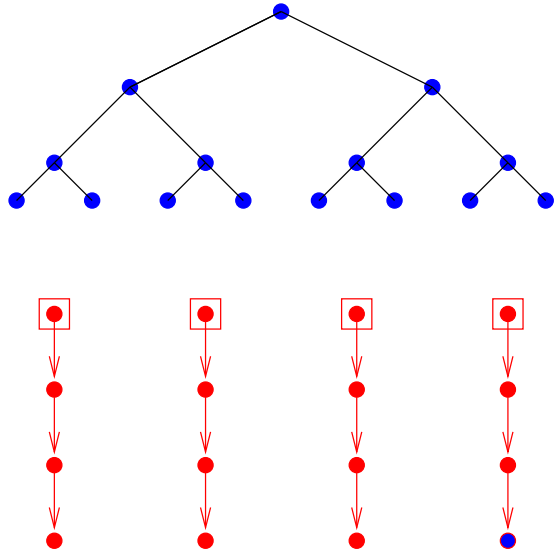

Selecting winners

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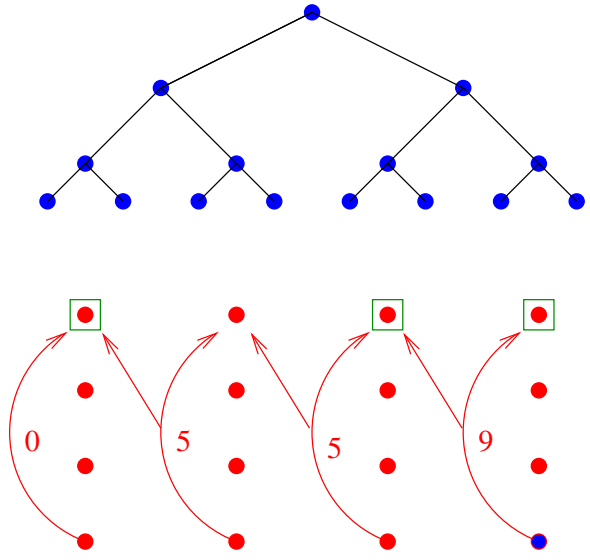
For each *rank*, the last leader with this *rank* becomes a *winner*.



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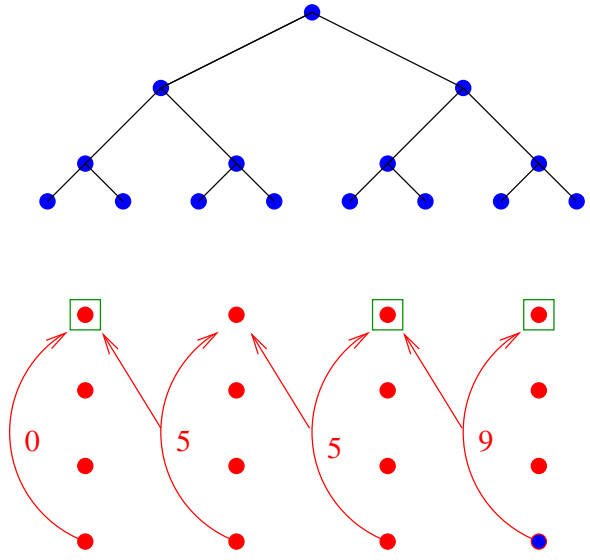

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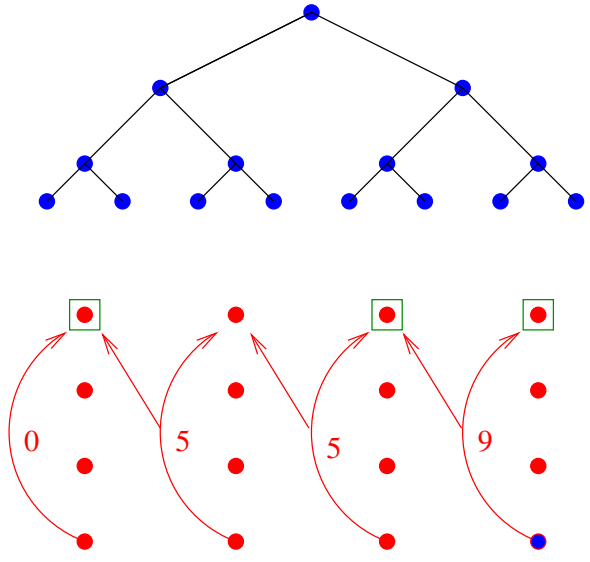
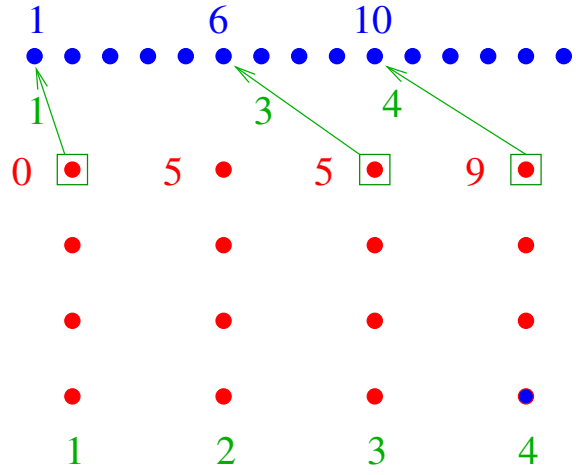
Time: $\lceil m/\lceil \lg(m+1) \rceil \rceil$



Setting group borders

Each *winner* informs its *successor* b_i in the *other sequence* about its *block number* in the i-th time slot.



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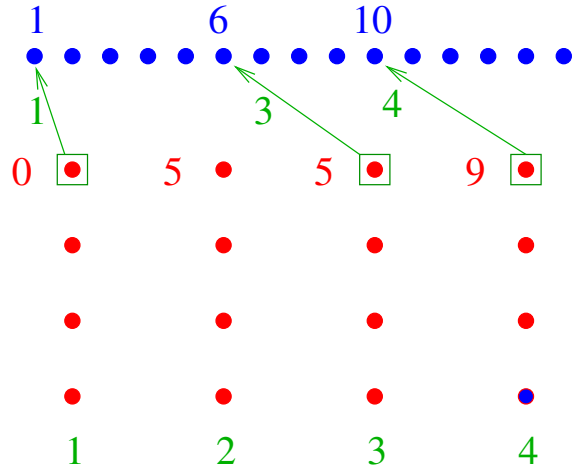

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Time: m.

Grouping

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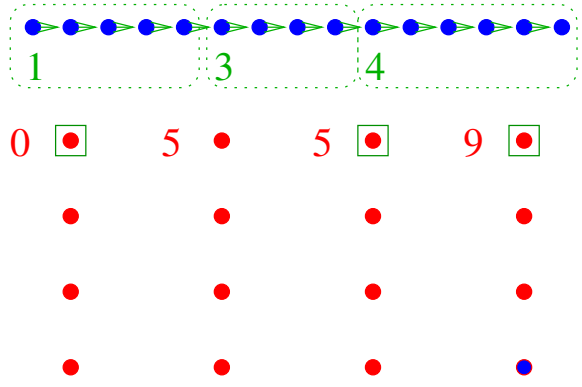
Successor ignores this if it was earlier informed by some *winner*.



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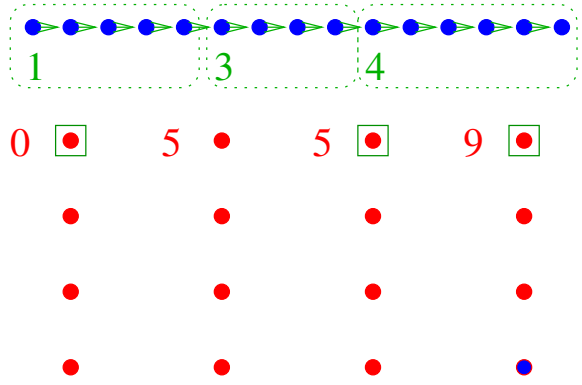
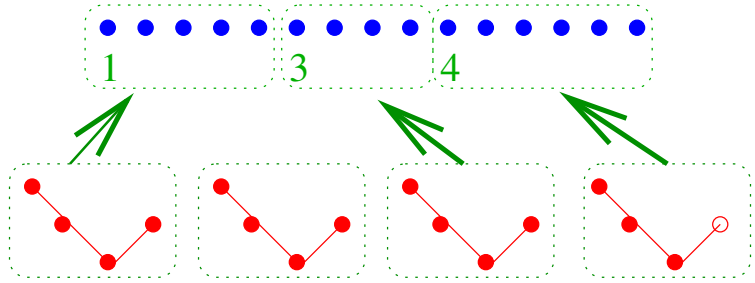


Energetic cost: 2.
Time: $m - 1$.

Ranking with energy $O(\lg \lg m)$

Each station listens only to the tree with its group number



- Energetic cost: listening: $O(\lg \lg m)$, broadcasting: 1.
- Time: $O(m)$

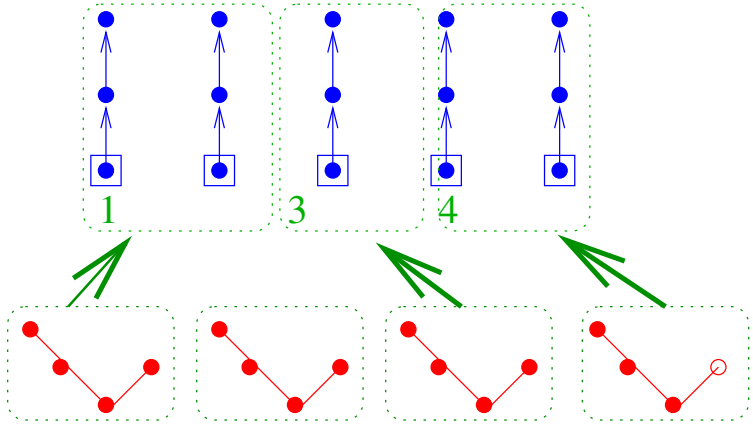
Iterating of regrouping

Instead of all stations computing their own ranks, blocks of $\approx \lg \lg m$ stations compute ranks of their *leaders*.

Now the density of the *leaders* is much higher.

The message to the next slave is:

(leader's key, **group***, rank, timer)*



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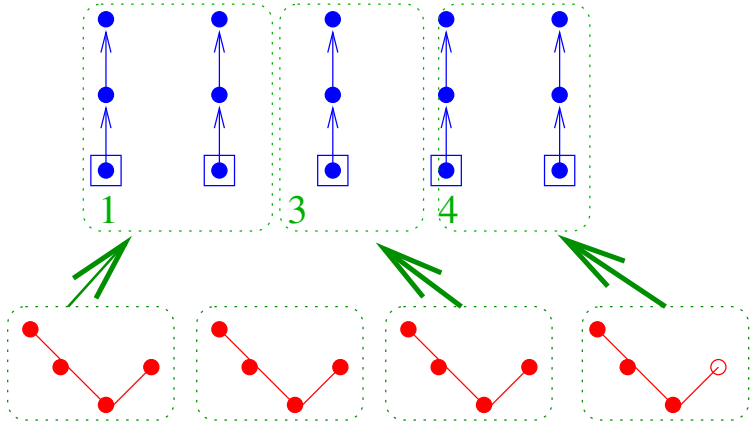

Then select the *winners* that split the other sequence into groups.
Note that each iteration *works in opposite direction* to the previous one.

Time and energy

For the whole iteration (*computing ranks of leaders, selecting winners, splitting the other sequence into groups*):

- the energetic cost is $O(1)$
- time is $O(m)$

$h(m,i)$ and $l(m)$

$$h(m,i) = \begin{cases} m, & \text{if } i = 0 \\ \lceil \lg(m+1) \rceil, & \text{if } i \geq 1 \end{cases}$$

$h(m, i+1)$ is the *height* of *balanced binary tree* with $h(m,i)$ nodes.
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$$l(m) = \min\{i : h(m,i) \leq 2\}.$$

$$\underbrace{h(m,0), h(m,1), \ldots 2}_{l(m)}, 2, 2, 2, 2 \ldots$$

Note that: $l(m)$ is $O(\lg^* m)$.

$O(\lg^* m)$

After $2\lceil l(m)/2 \rceil + 2 = O(\lg^* m)$ iterations the group numbers in each sequence refer to blocks of size two in the other sequence.
Hence, each station can compute its rank by listening to at most two messages.

$O(\lg^* m)$

After $2\lceil l(m)/2 \rceil + 2 = O(\lg^* m)$ iterations the group numbers in each sequence refer to blocks of size two in the other sequence.
Hence, each station can compute its rank by listening to at most two messages.

Theorem: Merging of two sequences of length m can be done in time $\Theta(m \lg^* m)$ with energetic cost (of both listening and broadcasting) $\Theta(\lg^* m)$

Concluding remarks

- Exact asymptotic bounds for energetic cost?
  - of merging (constant?)
  - of sorting (we ignore cost of internal computations and stations may have larger memory) ($\Omega(\lg n)$?)

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  - of merging (constant?)
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- The case: each station stores k keys instead of one (Merging algorithm in DELIS-TR-0239: energ. cost: $\approx 8k + 4\lg m$, time: $\approx 6m \cdot k + m$)

Thank you!