NONBLOCKER SET:

Parameterized Algorithmics for MinDS

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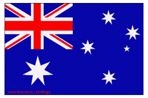
[2]The University of Newcastle 

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Overview

- $\mathcal{FPT}$: the methodology

- Problem definition & introductory example

- Mathematics off the shelf

- Algorithmic results

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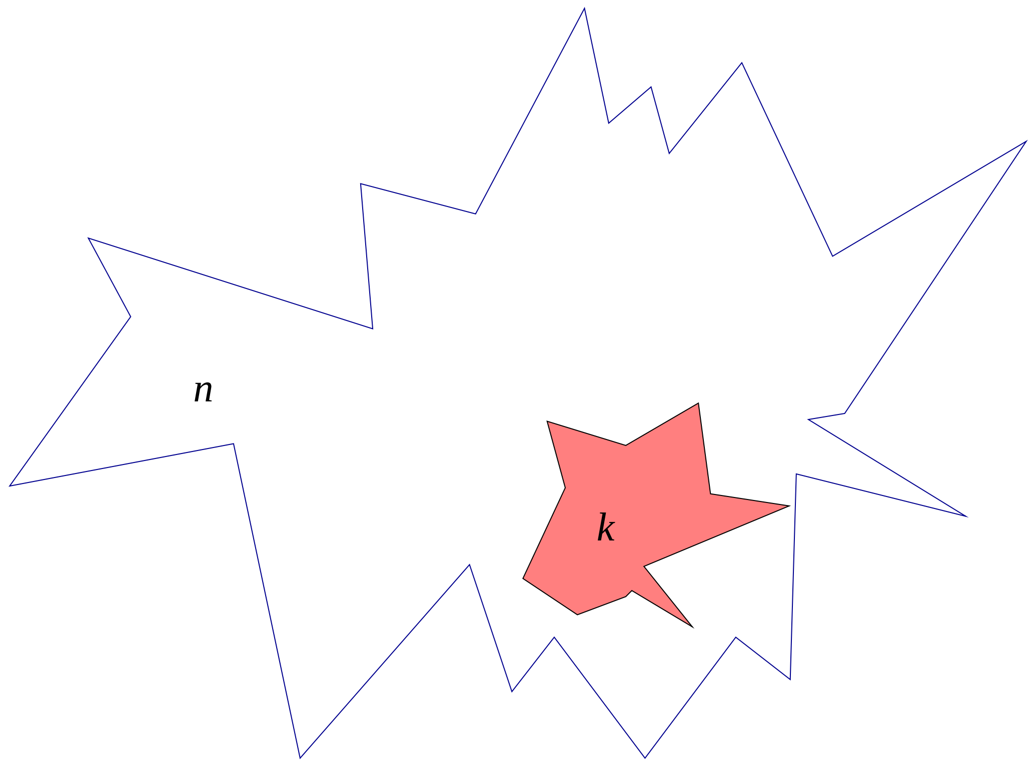
The Curse of Combinatorics



$\boxed{\text{Parameterized complexity in a nutshell}}$

Running time $\mathcal{O}(f(k)p(n))$

Problem kernel of size $g(k)$, computable in time $q(n)$.

Thm.: Both approaches yield the same.

Complexity class: $\mathcal{FPT}$

Standard approaches: search trees & kernelization

A **kernelization** for problem P maps an instance (I, k) of P onto another instance (I', k') of P

- in polynomial time, such that

- $|I'| + k' \leq g'(k)$ for a suitable function g'.

An upper bound $g(k)$ of $|I'|$ is called (problem) kernel size.
Standard size measure for graphs: # vertices n

Examples

Problem	Property
k-VC	kernel size $2k$ (NT)
$(n-k_d)$-VC or k_d-IS	W[1] complete
k-PVC	$2k$ kernel, no $< (4/3)k$ kernel unless $\mathcal{P} = \mathcal{NP}$
k_d-PIS	$4k_d$ kernel, no $< 2k_d$ kernel unless $\mathcal{P} = \mathcal{NP}$
k-DS	W[2] complete
$(n-k_d)$-DS	kernel size $(5/3)k_d + 3$ this paper
k-PDS	$67k$ kernel, no $< 2k$ kernel unless $\mathcal{P} = \mathcal{NP}$
$(n-k_d)$-PDS	kernel size $2k_d$ this paper

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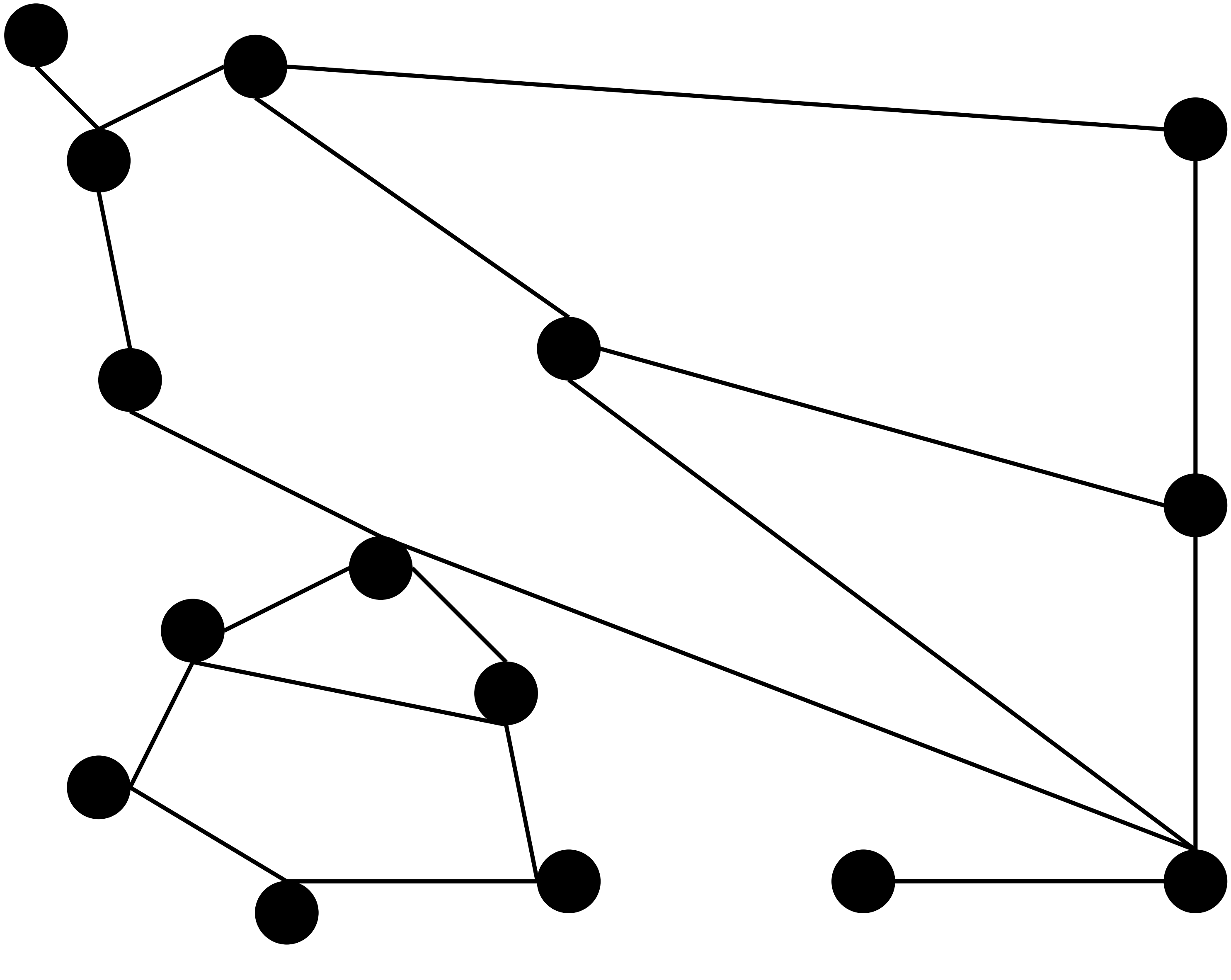
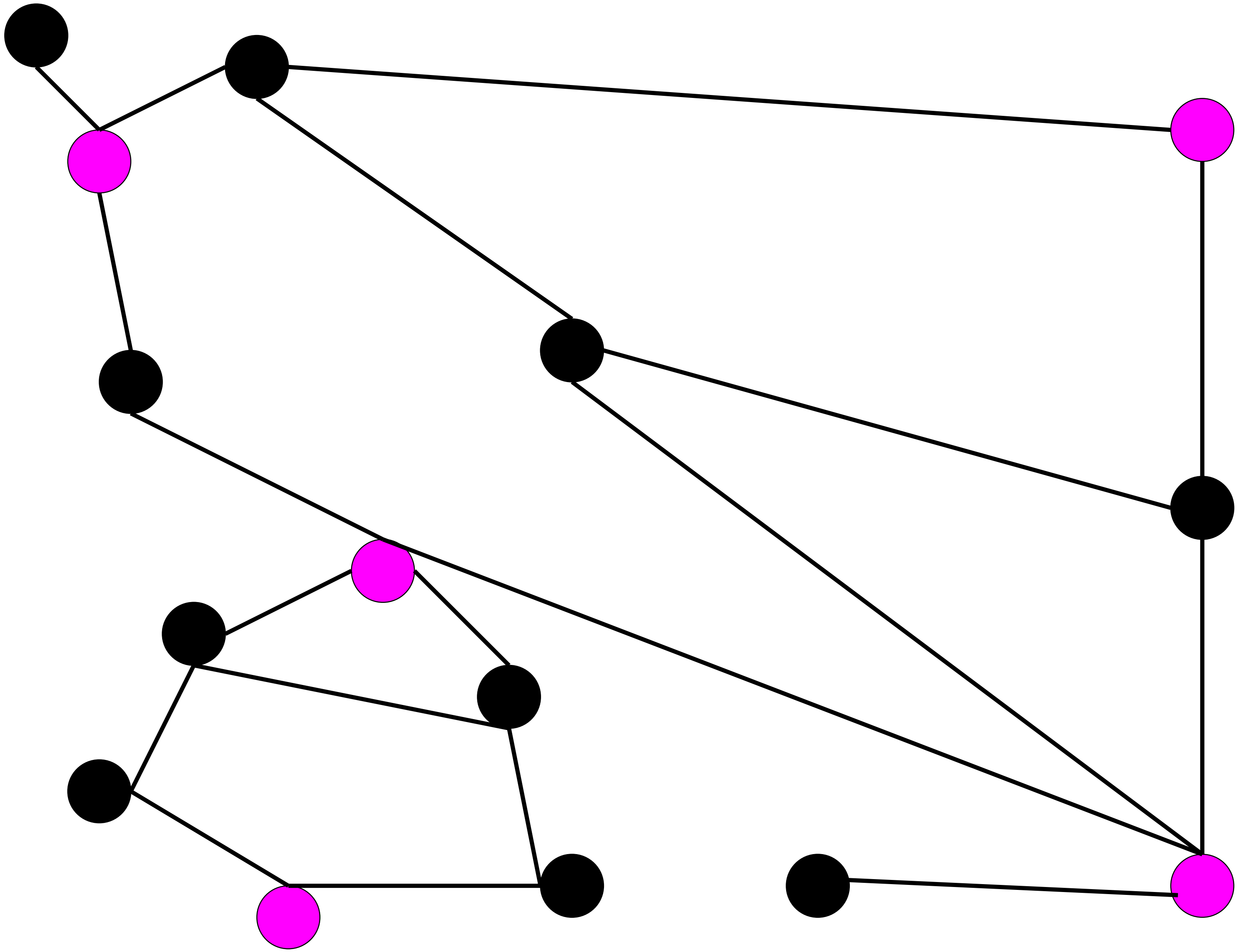
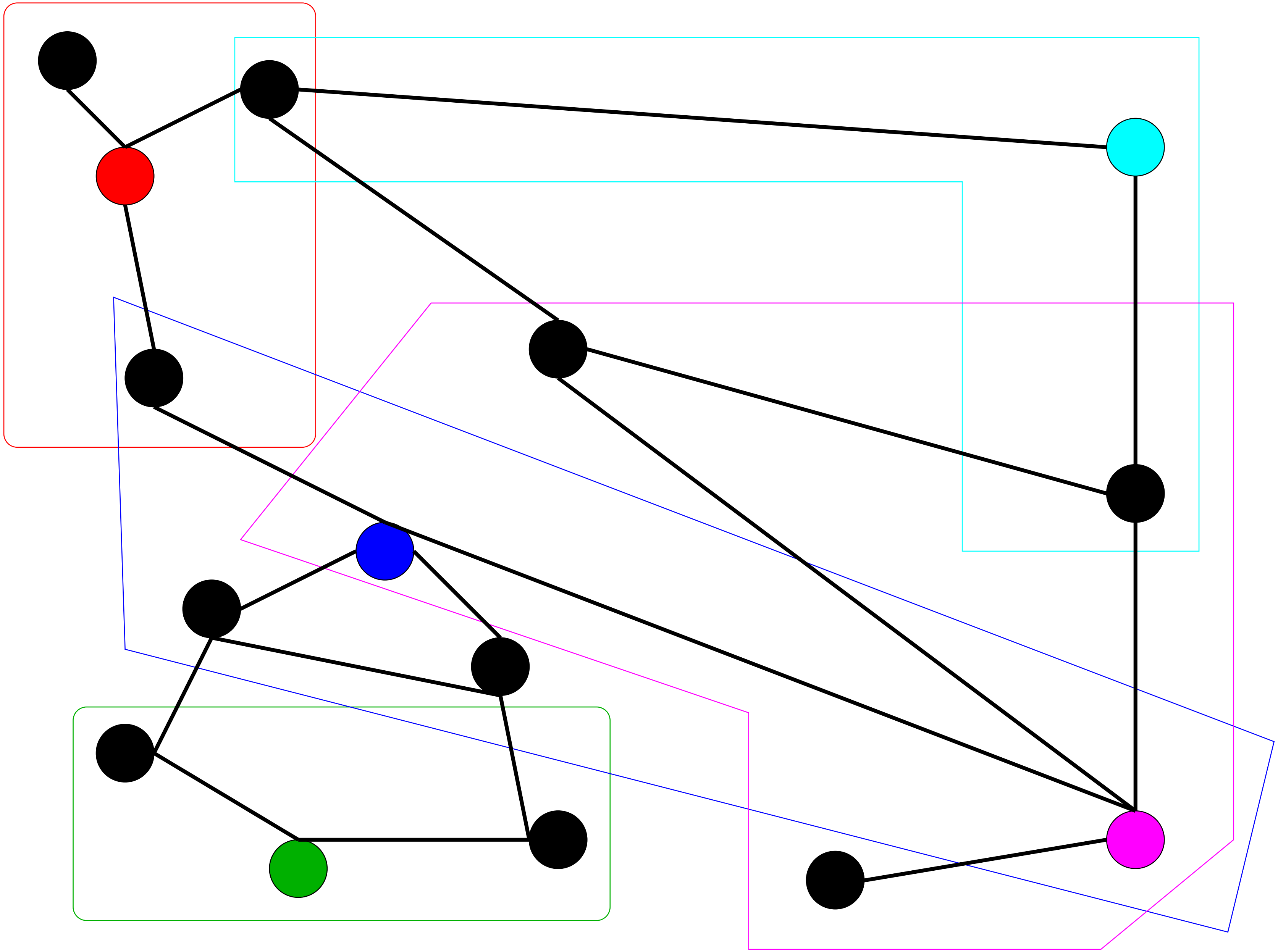
MINIMUM DOMINATING SET

A dominating set of $G = (V, E)$ is a vertex set D such that $N[D] = V$, where $N[x]$ is the closed neighborhood of x and $N[D] = \bigcup_{x \in D} N[x]$

Given: An undirected graph $G = (V, E)$
Task: Find a dominating set $D \subseteq V$ of minimum size.



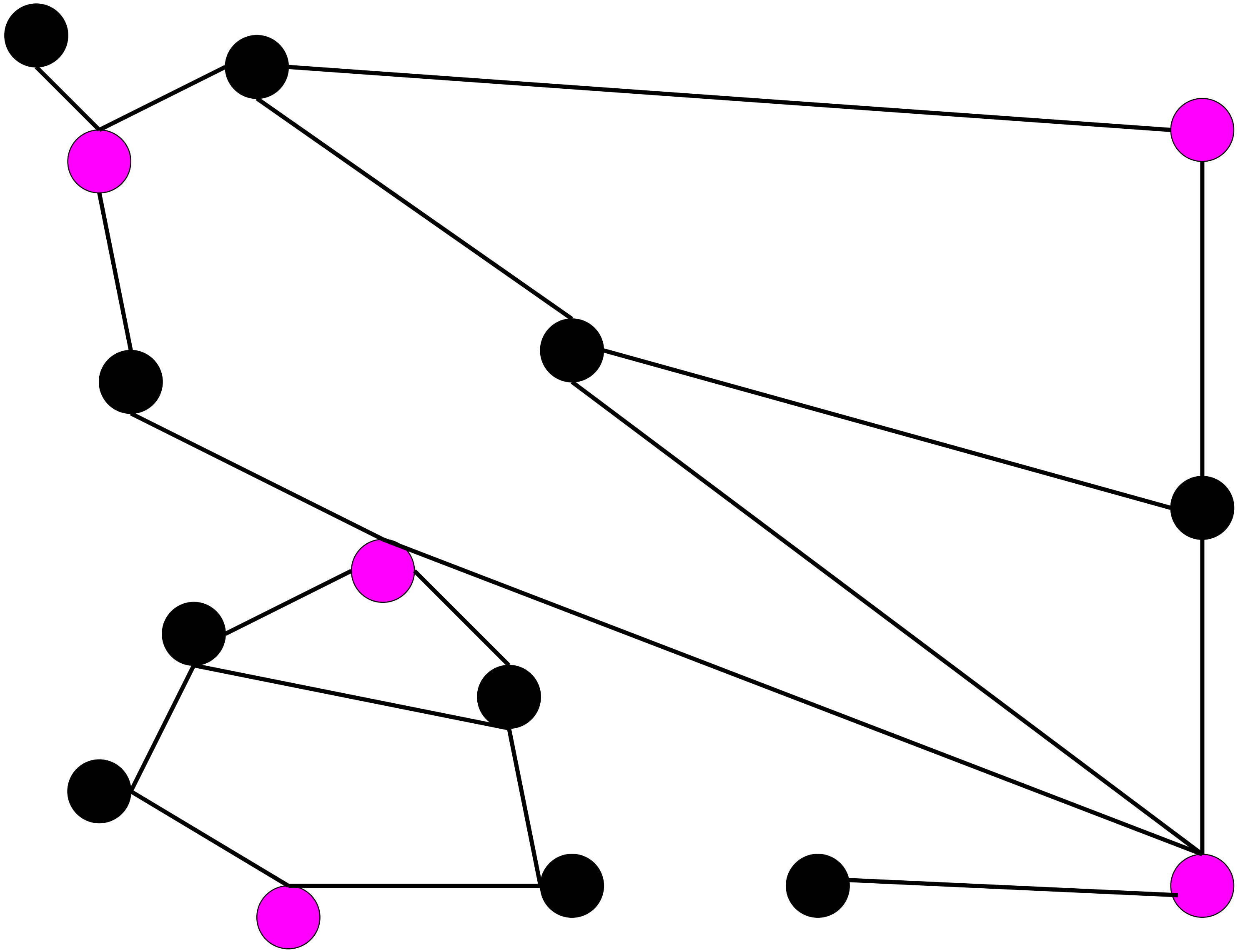




Our problem

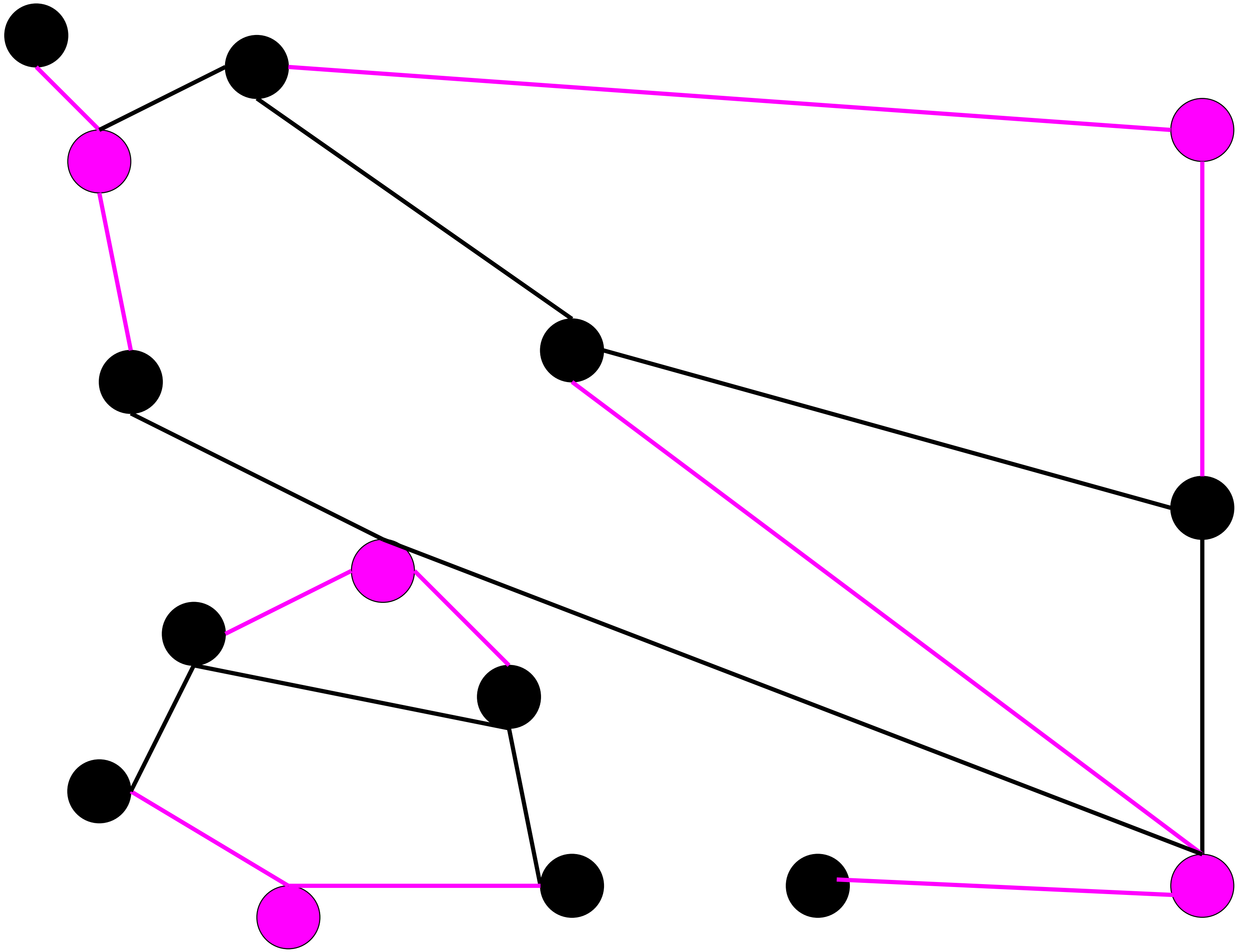
We call the complement of a dominating set a nonblocker set.

NONBLOCKER SET (NB)
Given: A graph $G = (V, E)$
Parameter: a positive integer k_d
Question: Is there a *nonblocker set* $N \subseteq V$ with $|N| \geq k_d$?



Equivalent characterization

Lemma: The size of the maximum nonblocker set of a graph equals the size of a maximum minimal edge cover set.



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An old result of Ore [1962]

Theorem 1 *If a connected graph* $G = (V, E)$ *has minimum degree one, then the size of its minimum dominating set is at most* $(1/2) \cdot |V|$*.*

Proof. G connected $\Rightarrow$ G has spanning tree $T = (V, E_T)$.
Fix a root $r \in V$ and
let V_{even} (V_{odd}) be the vertices of even (odd) distance from r in T.
Both V_{even} and V_{odd} are dominating sets of G.
The smaller of either has at most $(1/2) \cdot |V|$ many vertices. ∎

How to use mathematical results off the shelf

...the size of its minimum dominating set is at most $(1/2) \cdot n$
$\Rightarrow$ If $k \geq (1/2)n$, then YES
$\equiv$ the size of its maximum nonblocker set is at least $(1/2) \cdot n$
$\Rightarrow$ If $k_d \leq (1/2)n$, then YES
OTHERWISE: $n < 2k_d$.

kernel found !?

Problem: If a connected graph $G = (V, E)$ has minimum degree one, ...

An improved result of Blank and McCuaig/Shepherd [1973/1989]

Theorem 2 *If a connected graph $G = (V, E)$ has minimum degree two and is not one of seven exceptional graphs (each of them having at most seven vertices), then the size of its minimum dominating set is at most $2/5 \cdot |V|$.*

Even better: a theorem of Reed

Theorem 3 *If a connected graph $G = (V, E)$ has minimum degree three, then the size of its minimum dominating set is at most $3/8 \cdot |V|$.*

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Using the Theorem of Ore

Reduction rule 1 *Delete isolated vertices (without changing the parameter).*

$\Rightarrow$ Kernelization algorithm:

1. Apply reduction rule.

2. If reduced graph has at least $2k_d$ vertices, YES.

3. Otherwise: We have a graph instance with less than $2k_d$ vertices.

How to get rid of vertices of degree one ?

NONBLOCKER SET WITH CATALYTIC VERTEX (NBCAT)

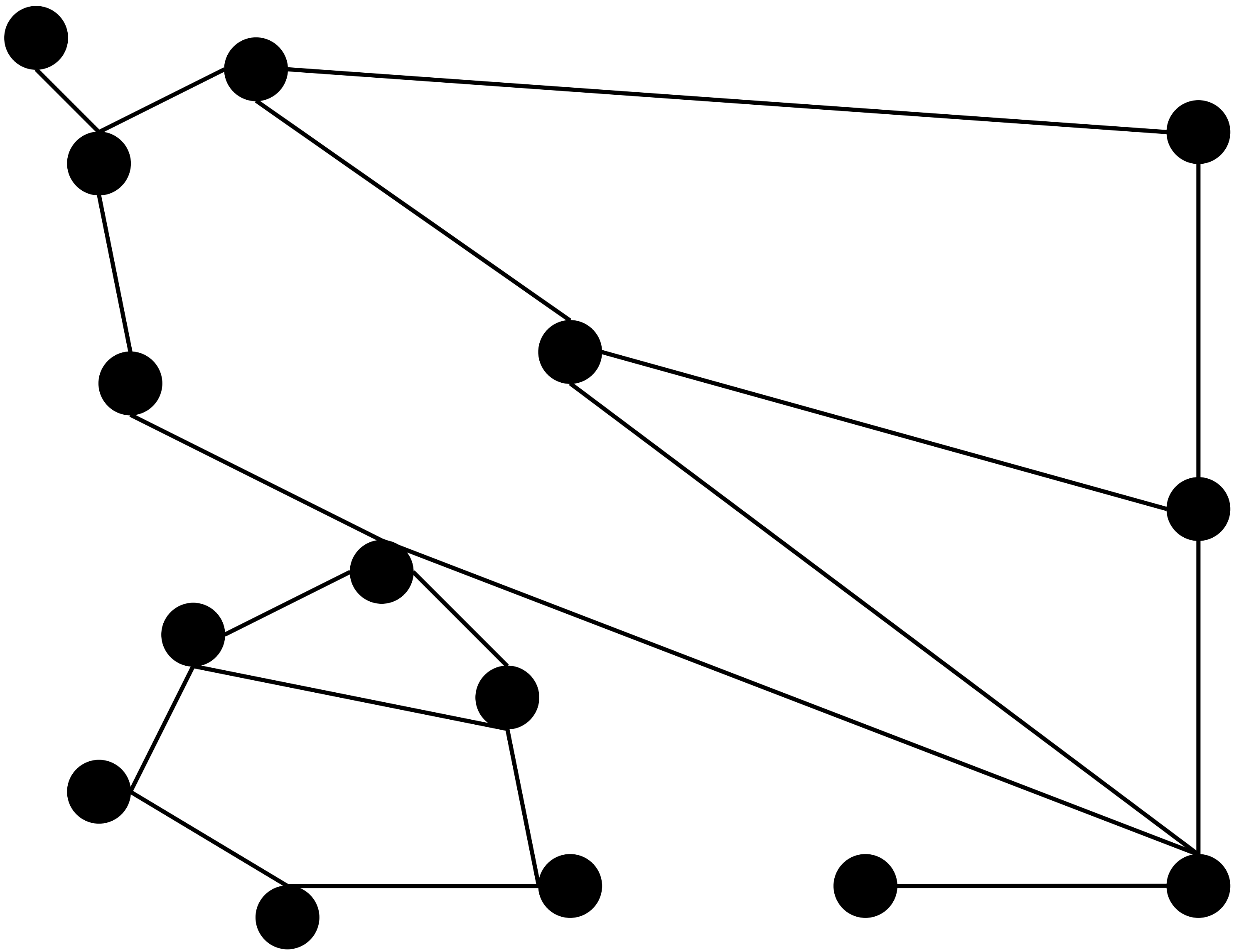
Given: A graph $G = (V, E)$, a catalytic vertex c

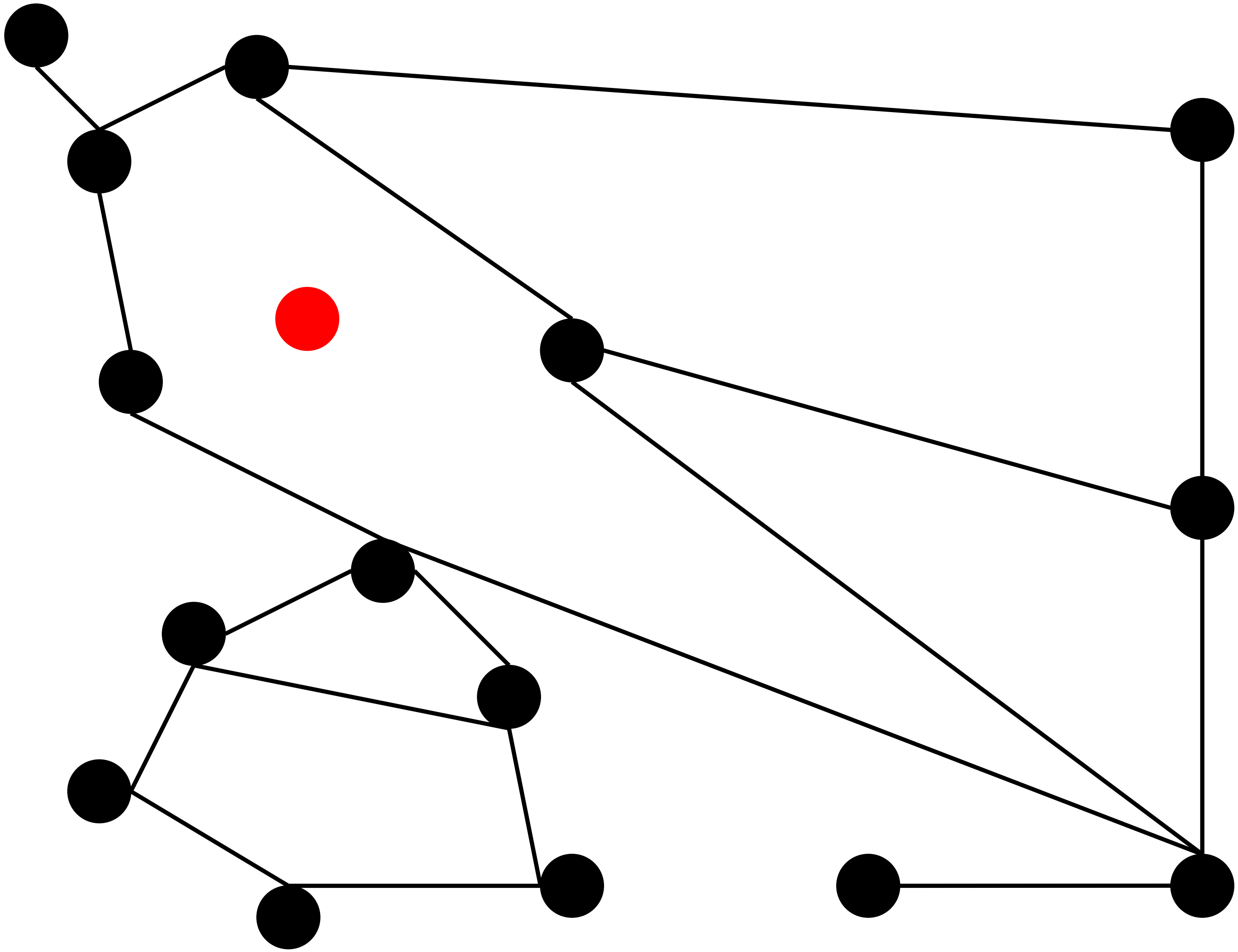
Parameter: a positive integer k_d

Question: Is there a *nonblocker set* $N \subseteq V$ with $|N| \geq k_d$ such that $c \notin N$?

How to introduce the catalyst

Reduction rule 2 (Catalyzation rule) *If* (G, k_d) *is a* NONBLOCKER SET*-instance with* $G = (V, E)$*, then* (G', c, k_d) *is an equivalent instance of* NONBLOCKER SET WITH CATALYTIC VERTEX*, where* $c \notin V$ *is a new vertex, and* $G' = (V \cup \{c\}, E)$*.*





How to use the catalyst: complete components

Reduction rule 3 (The Isolated Vertex Rule) *Let* (G, c, k_d) *be an instance of* NBCAT. *If* C *is a complete graph component of* G *that does not contain* c*, then reduce to* $(G - C, c, k_d - (|C| - 1))$.

Notice that this way isolated vertices are eliminated.

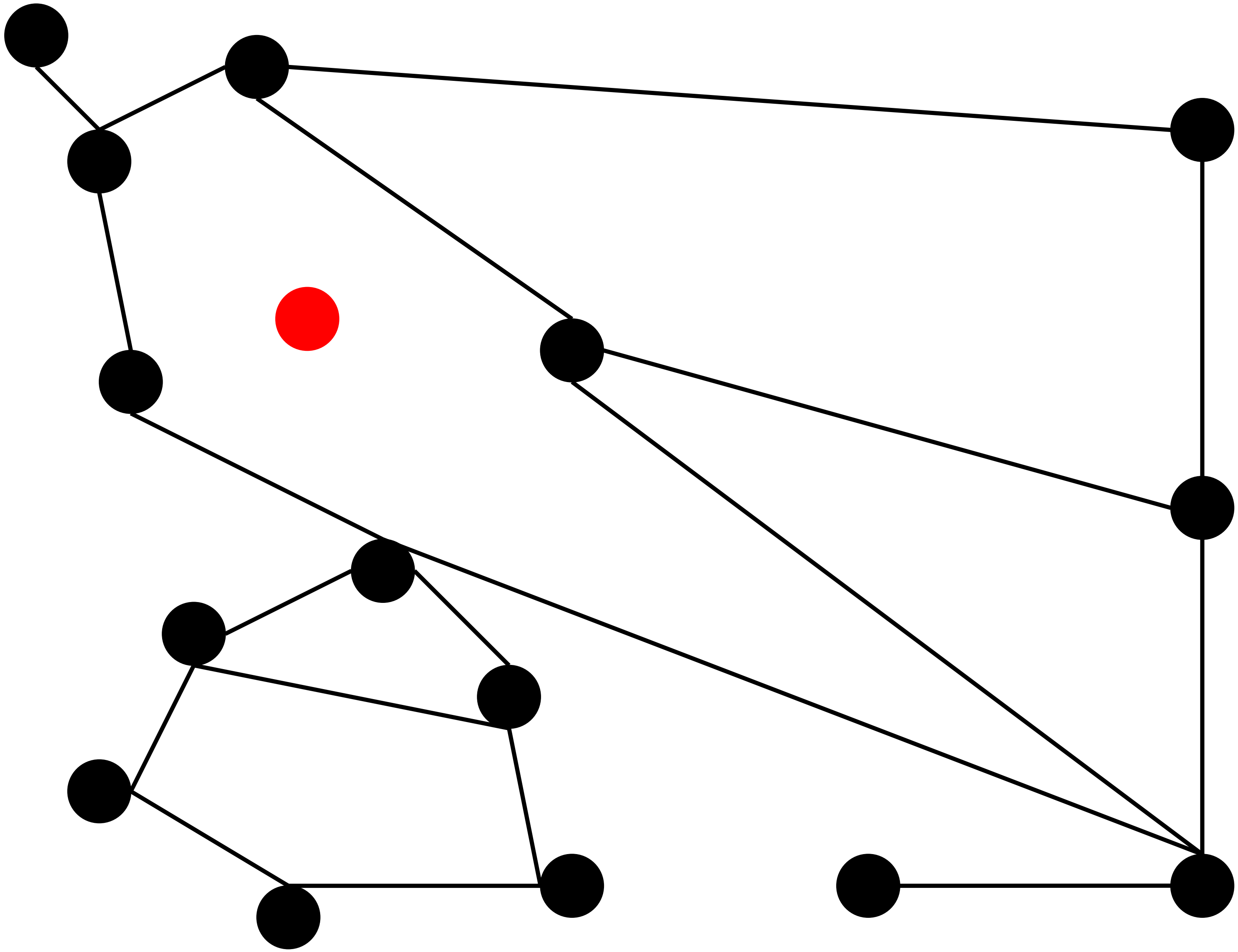
How to use the catalyst: degree one vertices

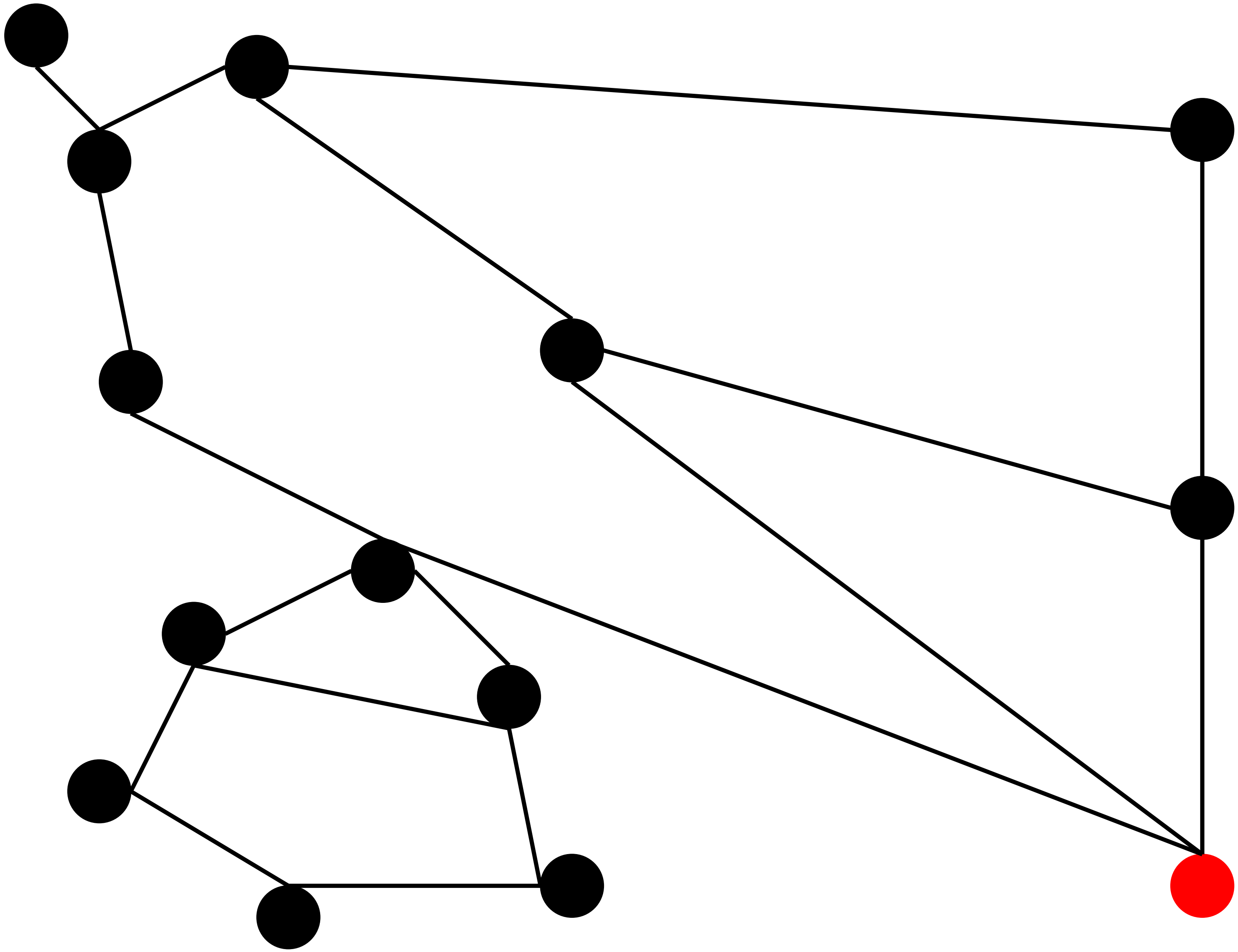
Merge catalyst with vertices that should be in the dominating set.

Reduction rule 4 (The Catalytic Rule) *Let* (G, c, k_d) *be an instance of* NON-BLOCKER SET WITH CATALYTIC VERTEX. *Let* $v \neq c$ *be a vertex of degree one in* G *with* $N(v) = u$. *Transform* (G, c, k_d) *into* $(G', c', k_d - 1)$, *where:*

- *If* $u \neq c$ *then* $G' = G_{[c \leftrightarrow u]} \setminus v$, *i.e.,* G' *is the graph obtained by deleting* v *and merging* u *and* c *into a new catalytic vertex* $c' = \langle c \leftrightarrow u \rangle$.

- *If* $u = c$ *then* $G' = G \setminus v$ *and* $c' = c$.



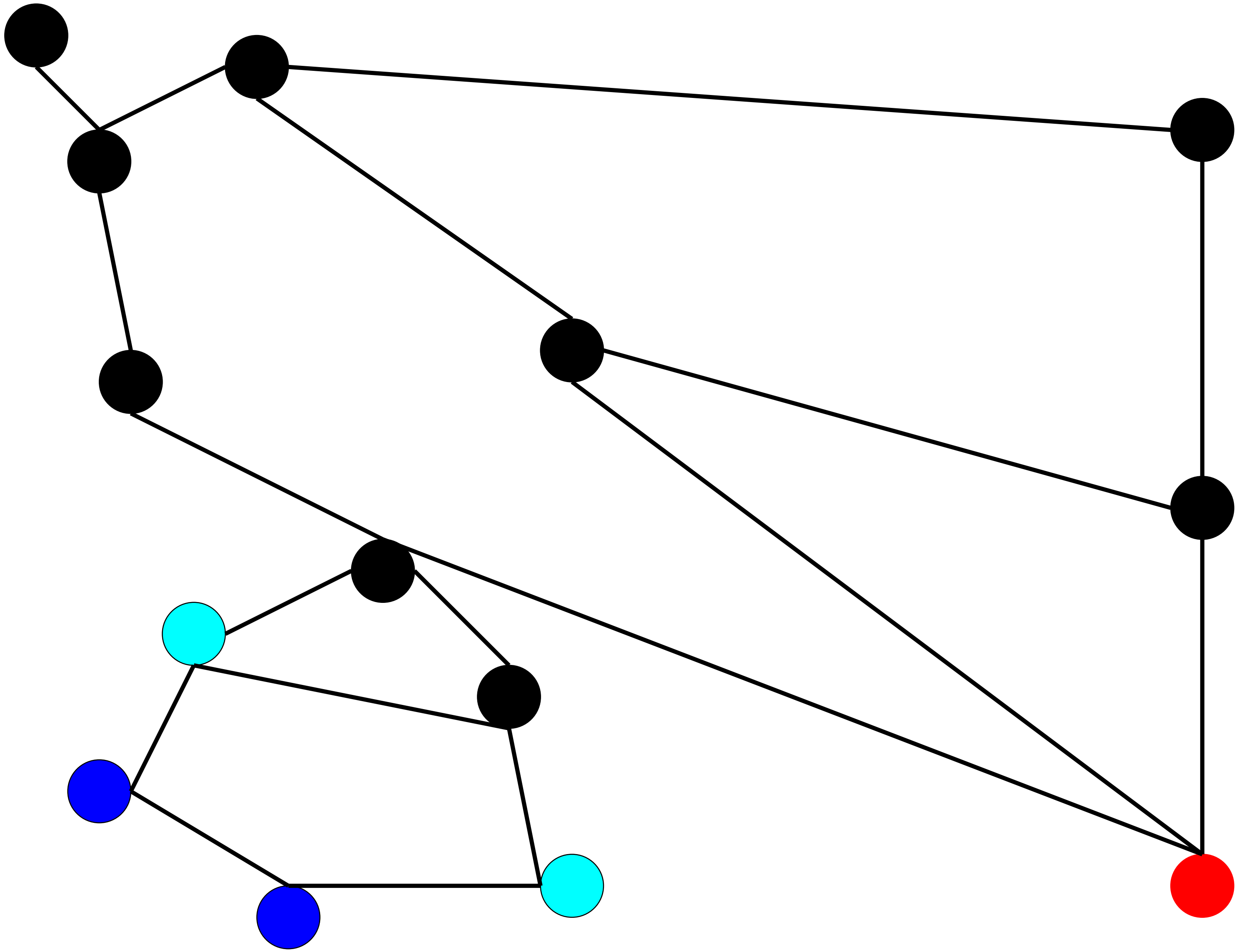


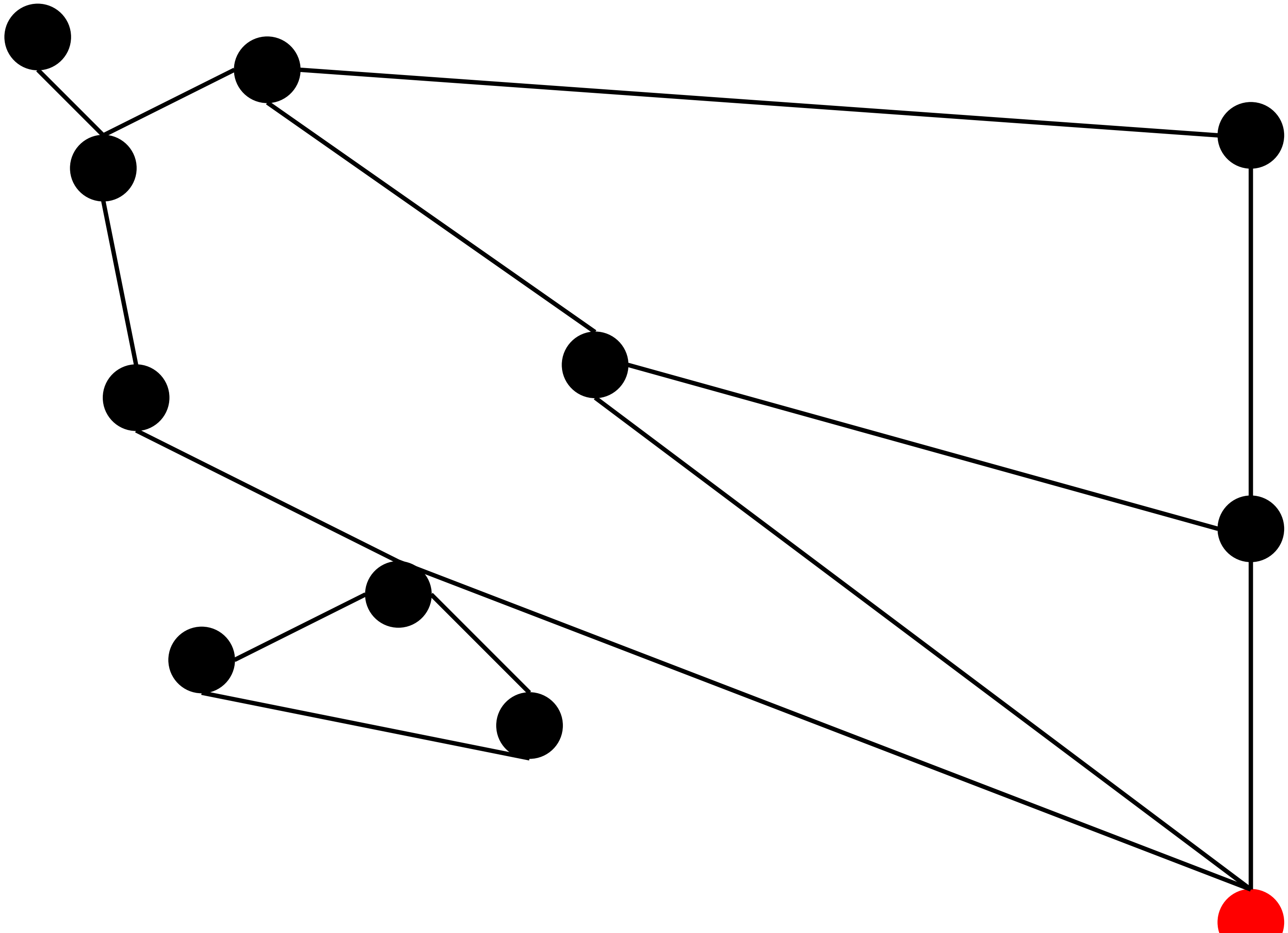
How to use the catalyst: neighbored degree two vertices I

Reduction rule 5 (The Degree Two Rule) *Let* (G, c, k_d) *be an instance of* NBCAT*. Let* u, v *be two vertices of degree two in* G *such that* $u \in N(v)$ *and* $|N(u) \cup N(v)| = 4$*, i.e.,* $N(u) = \{u', v\}$ *and* $N(v) = \{v', u\}$ *for some* $u' \neq v'$*.*

If $c \notin \{u, v\}$*, then merge* u' *and* v' *and delete* u *and* v *to get* $(G', c', k_d - 2)$*.*

If u' *or* v' *happens to be* c*, then* c' *is the merger of* u' *and* v'*; otherwise,* $c' = c$*.*

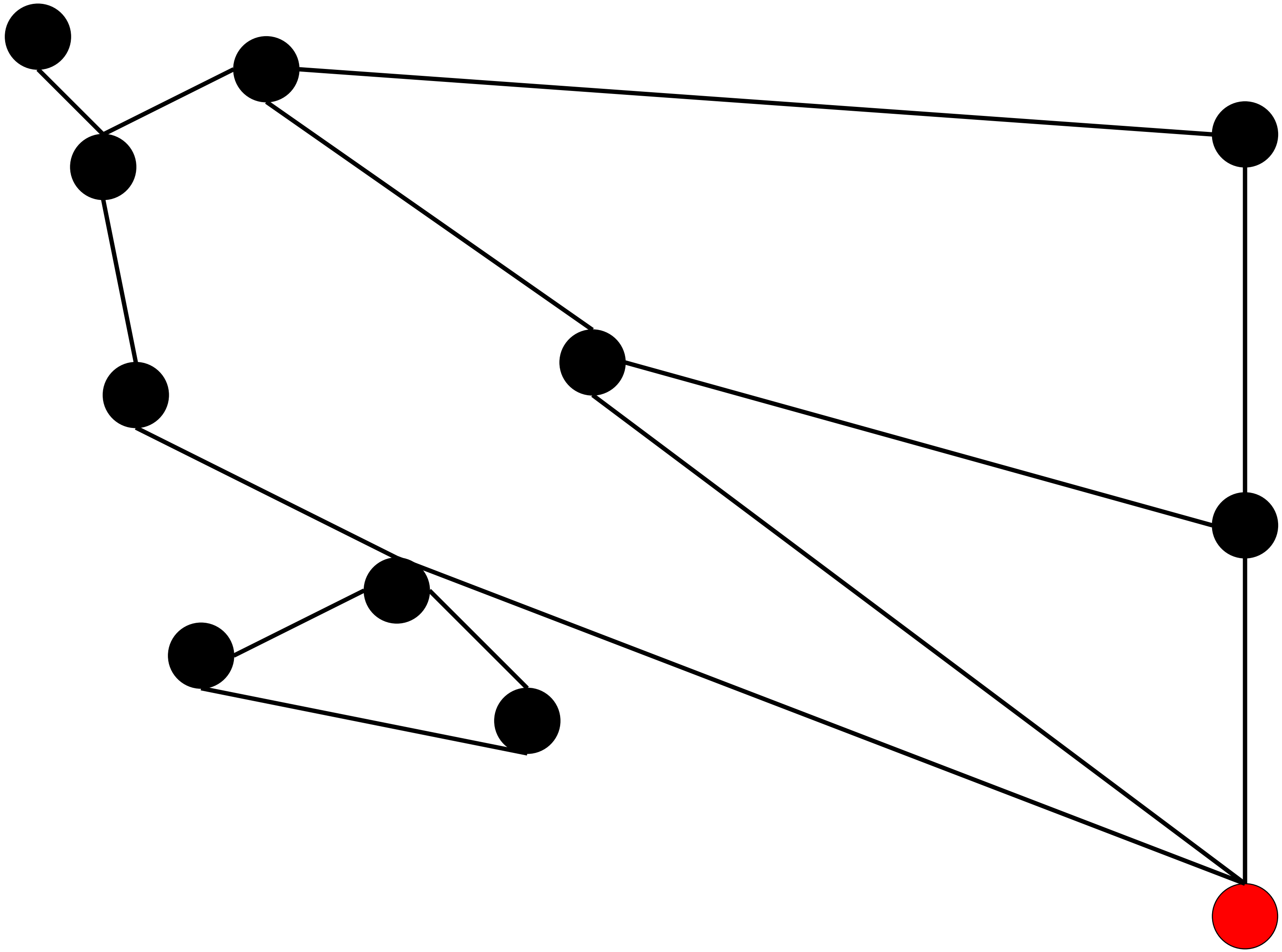


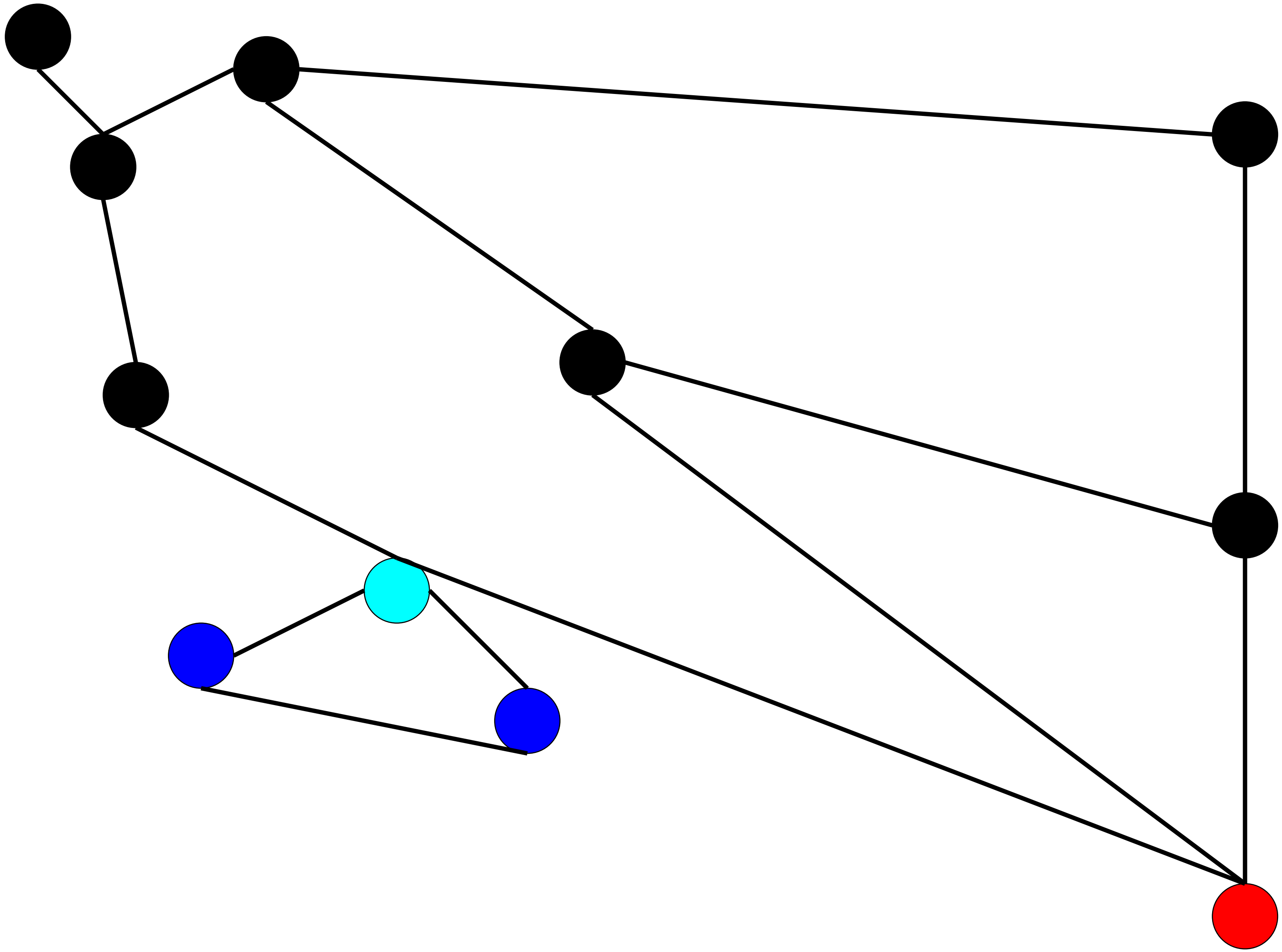


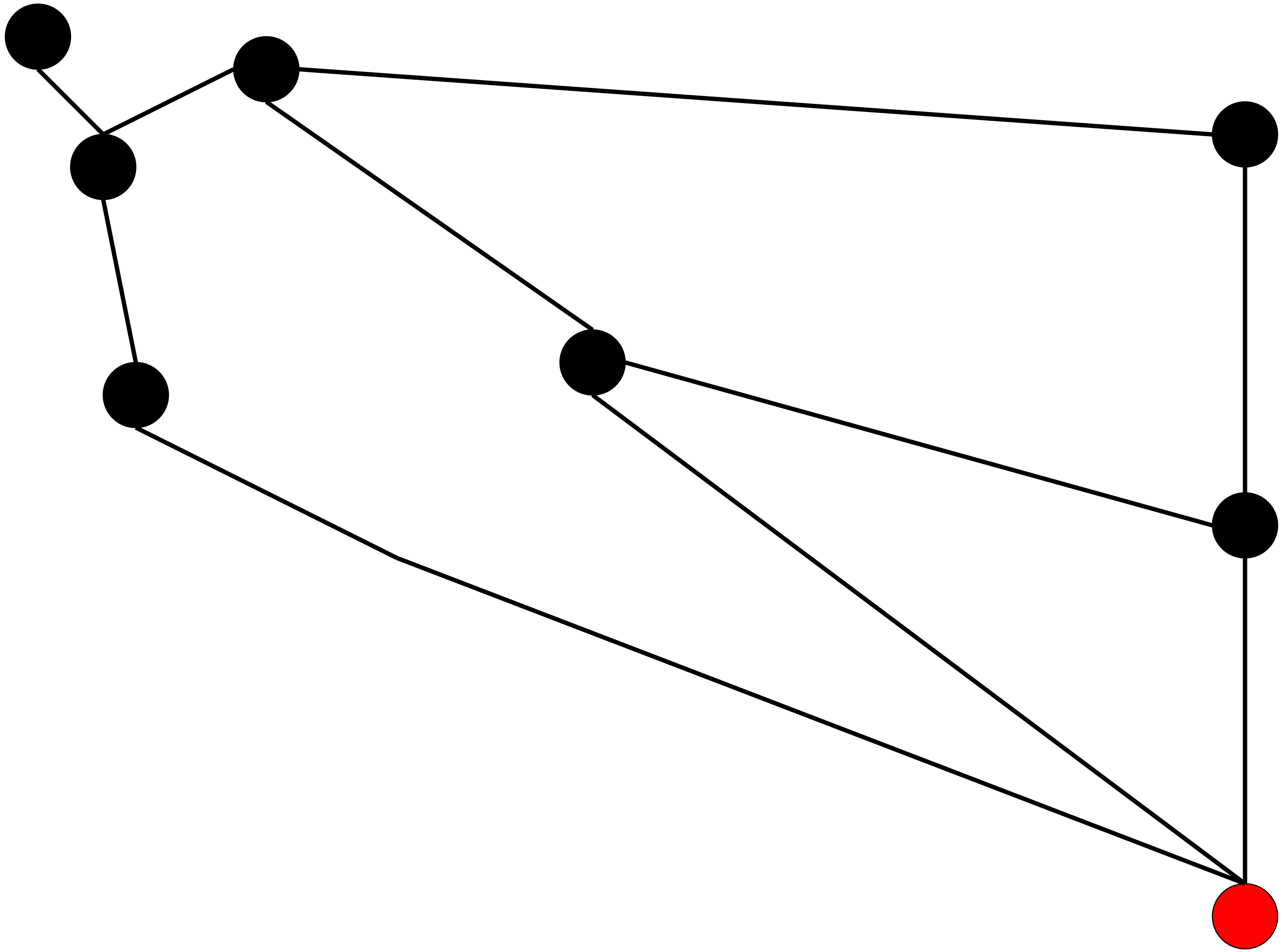
How to use the catalyst: neighbored degree two vertices II

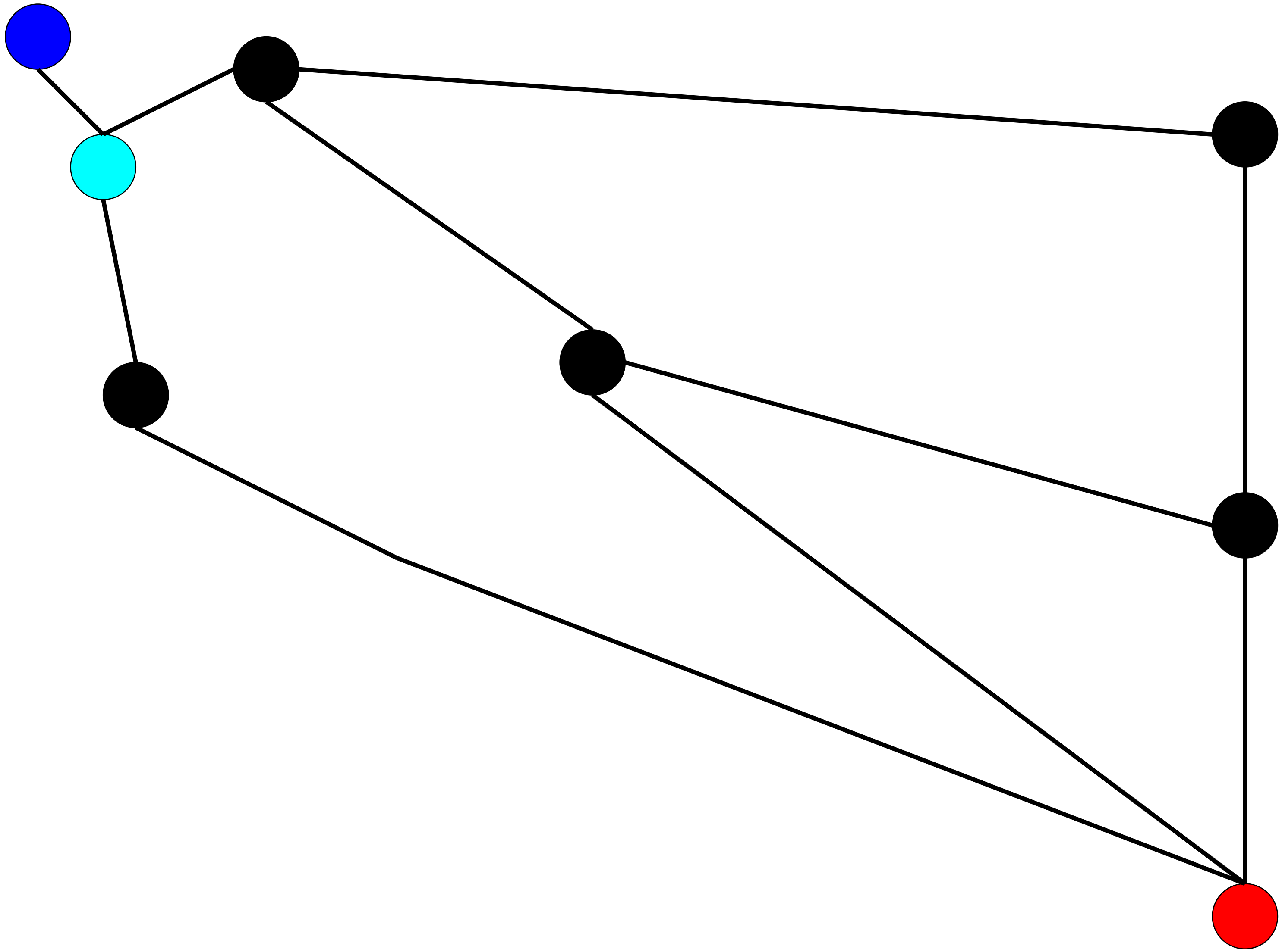
Generalize the degree-1-rule.

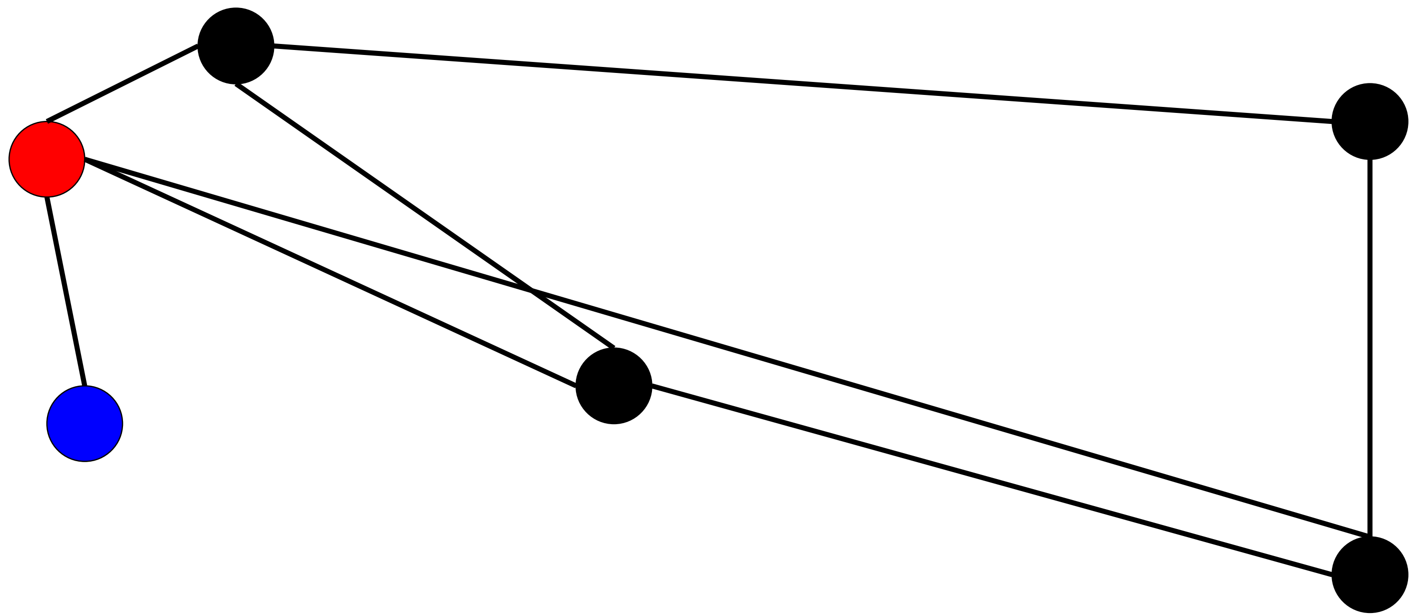
Reduction rule 6 (The Small Degree Rule) *Let* (G, c, k_d) *be an instance of* NBCAT*. Whenever you have a vertex* $x \in V(G)$ *whose neighborhood contains a non-empty subset* $U \subseteq N(x)$ *such that* $N(U) \subseteq U \cup \{x\}$ *and* $c \notin U$*, then you can merge* x *with the catalytic vertex* c *and delete* U *(and reduce the parameter by* $|U|$*).*

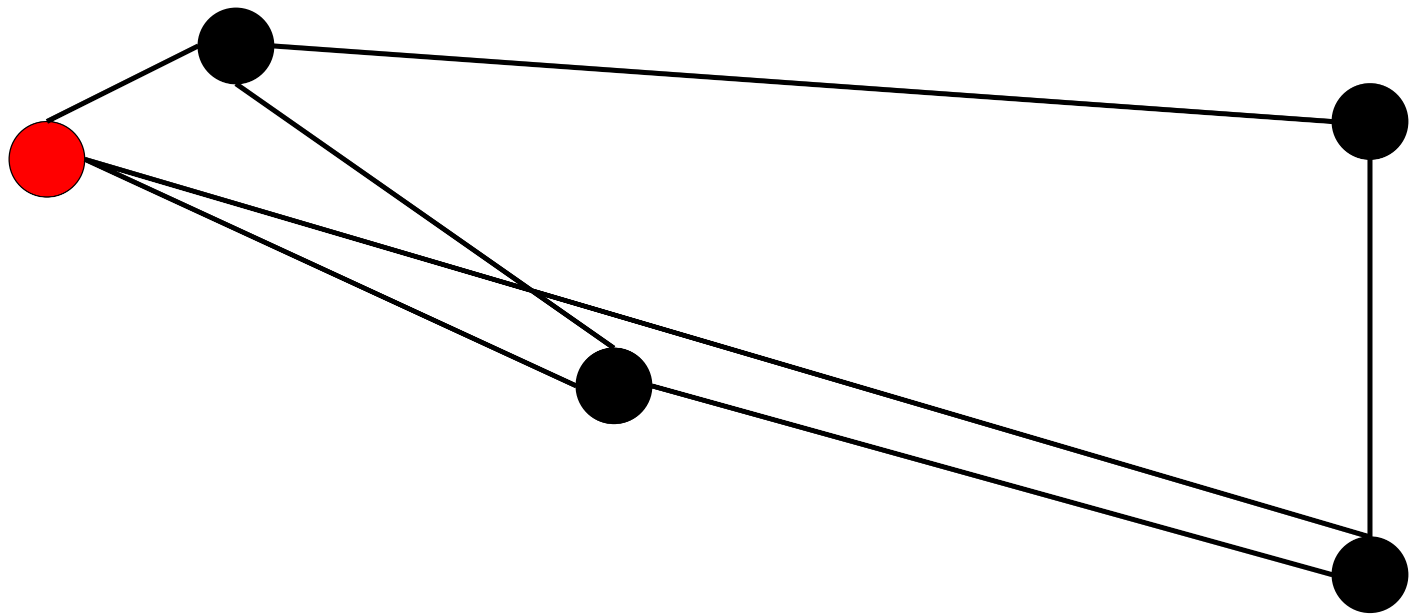


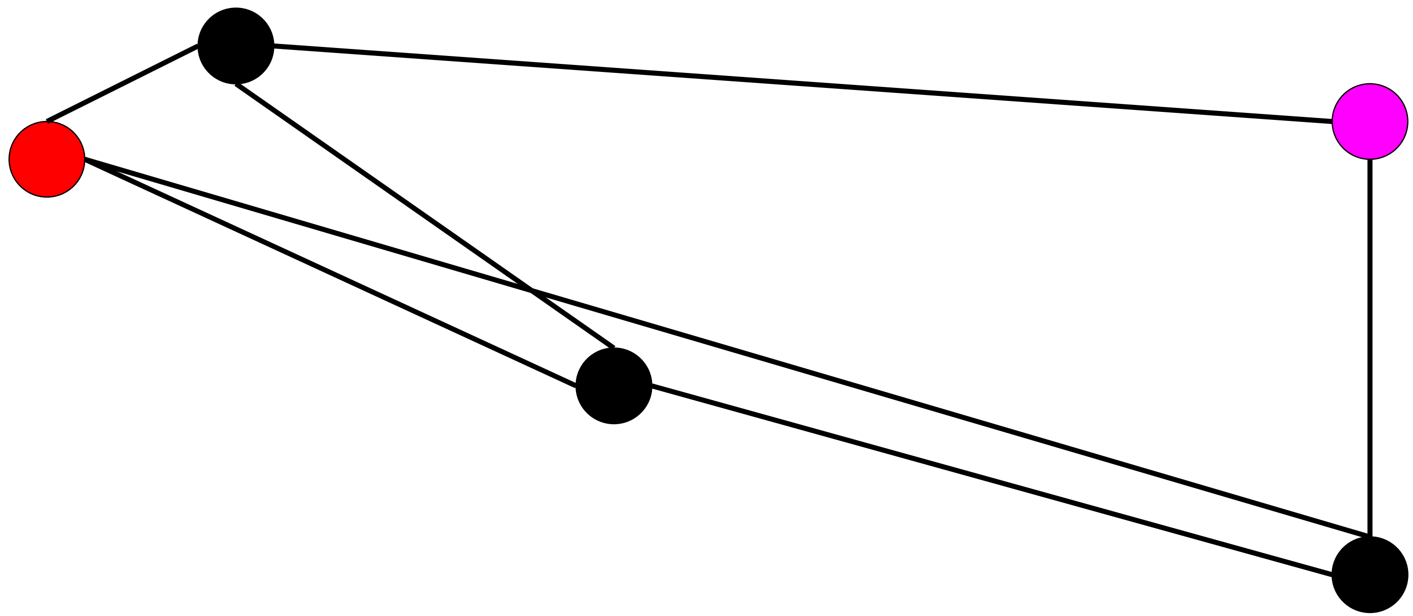


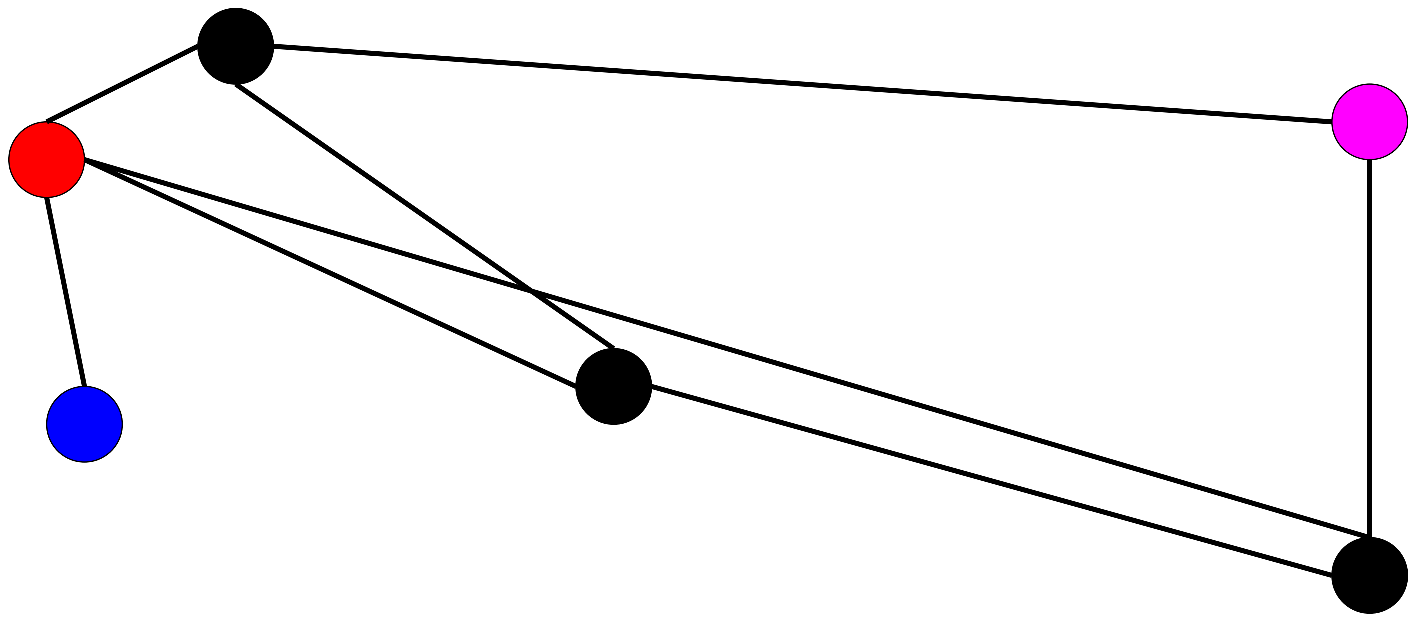


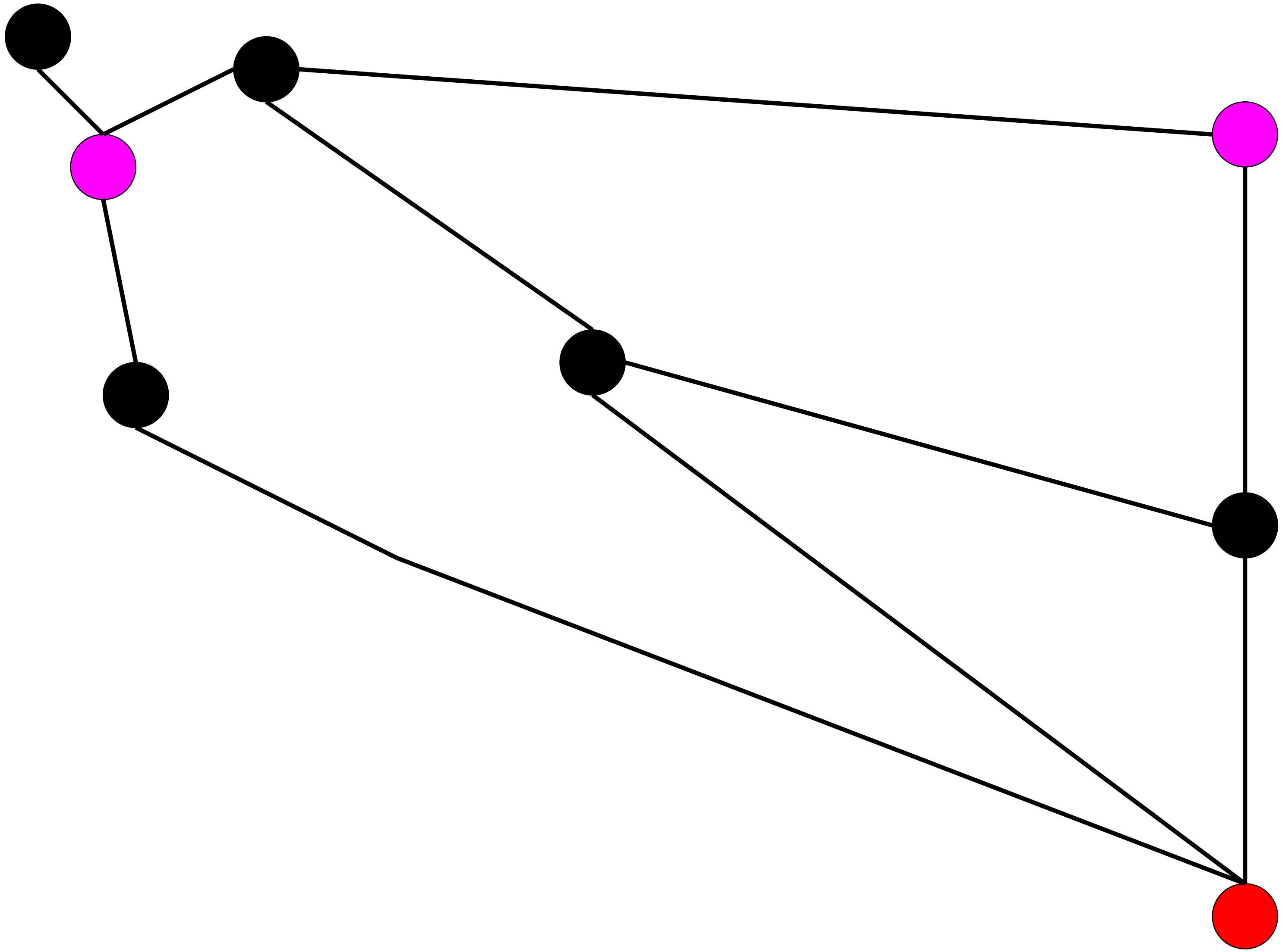


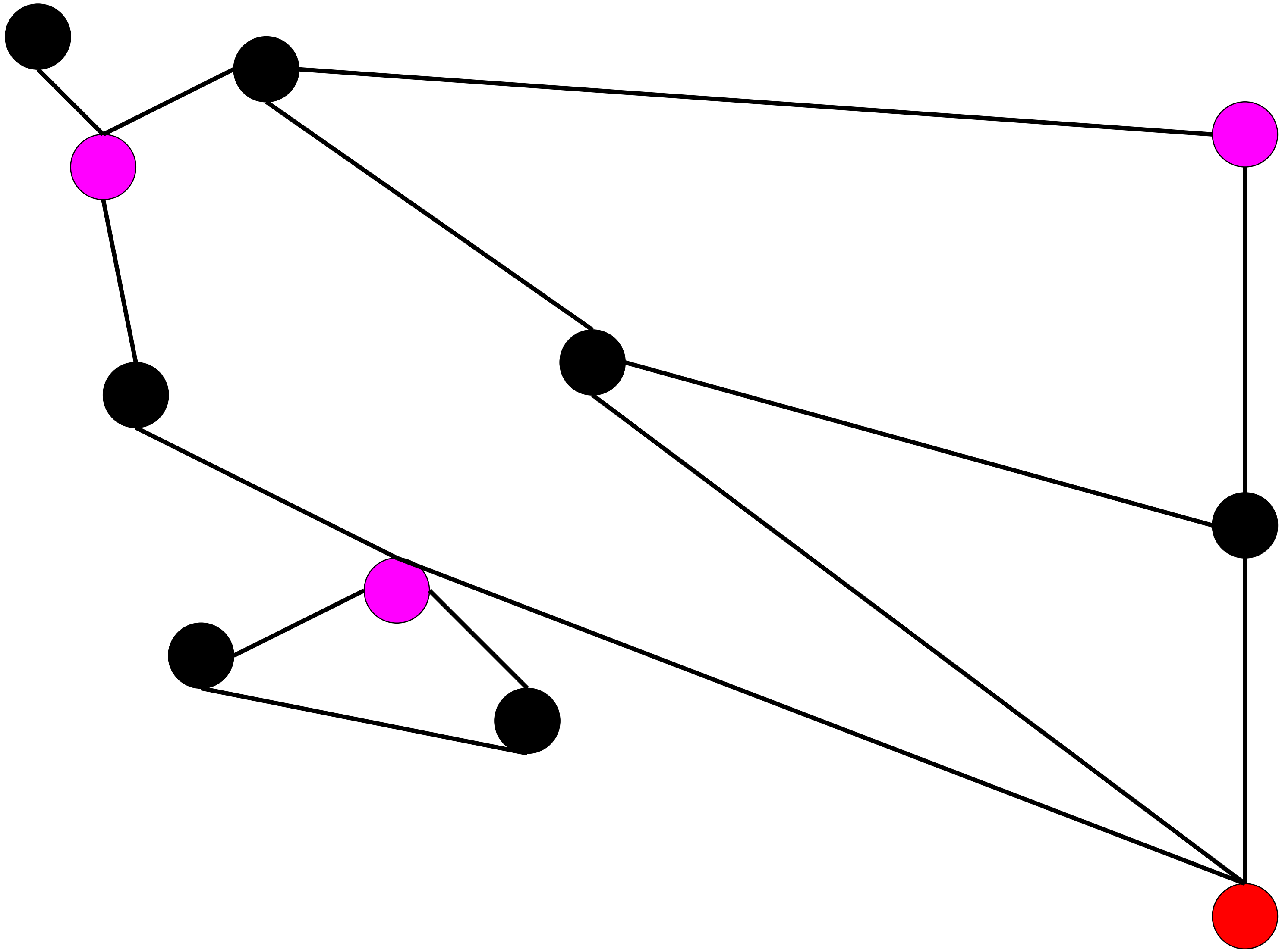


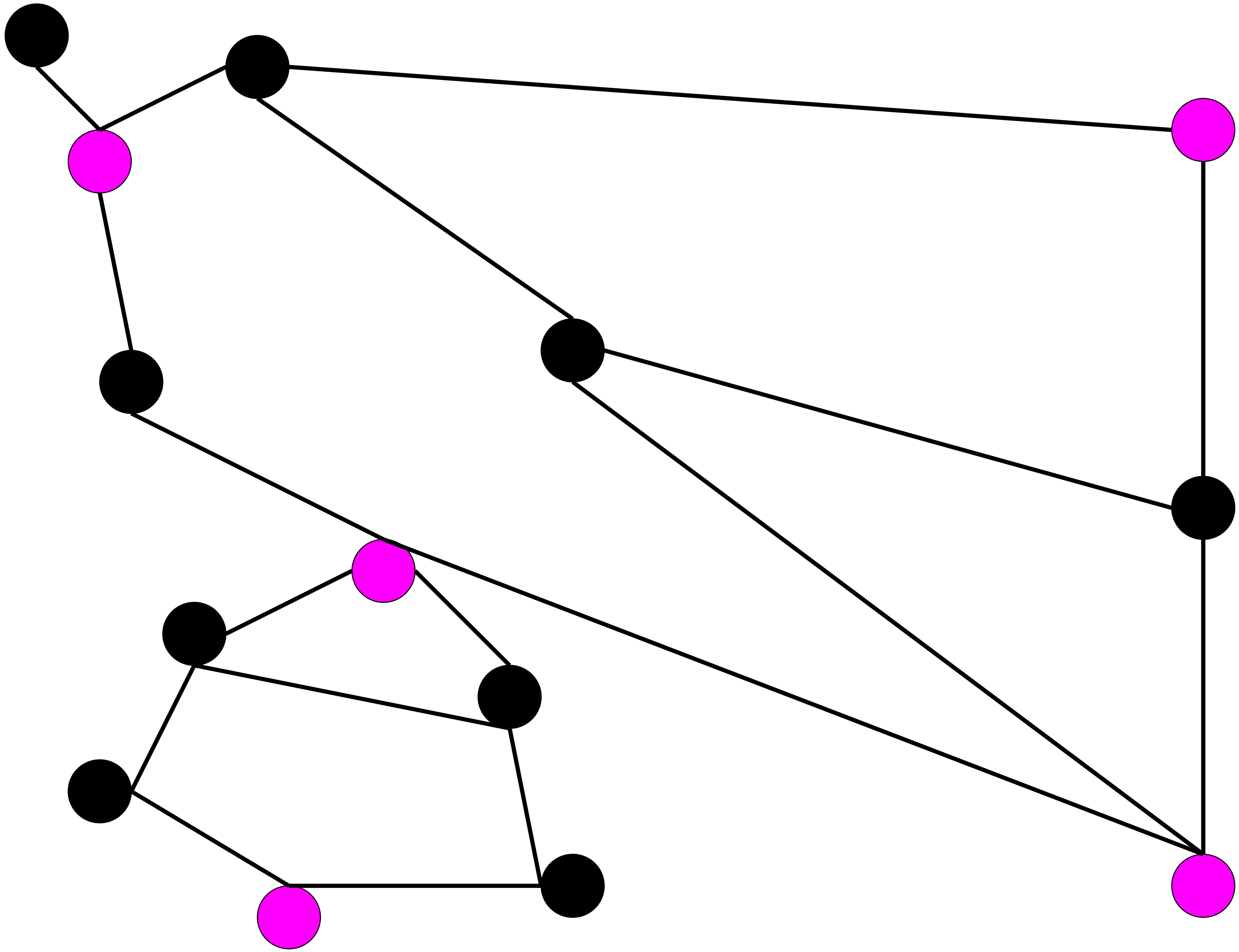


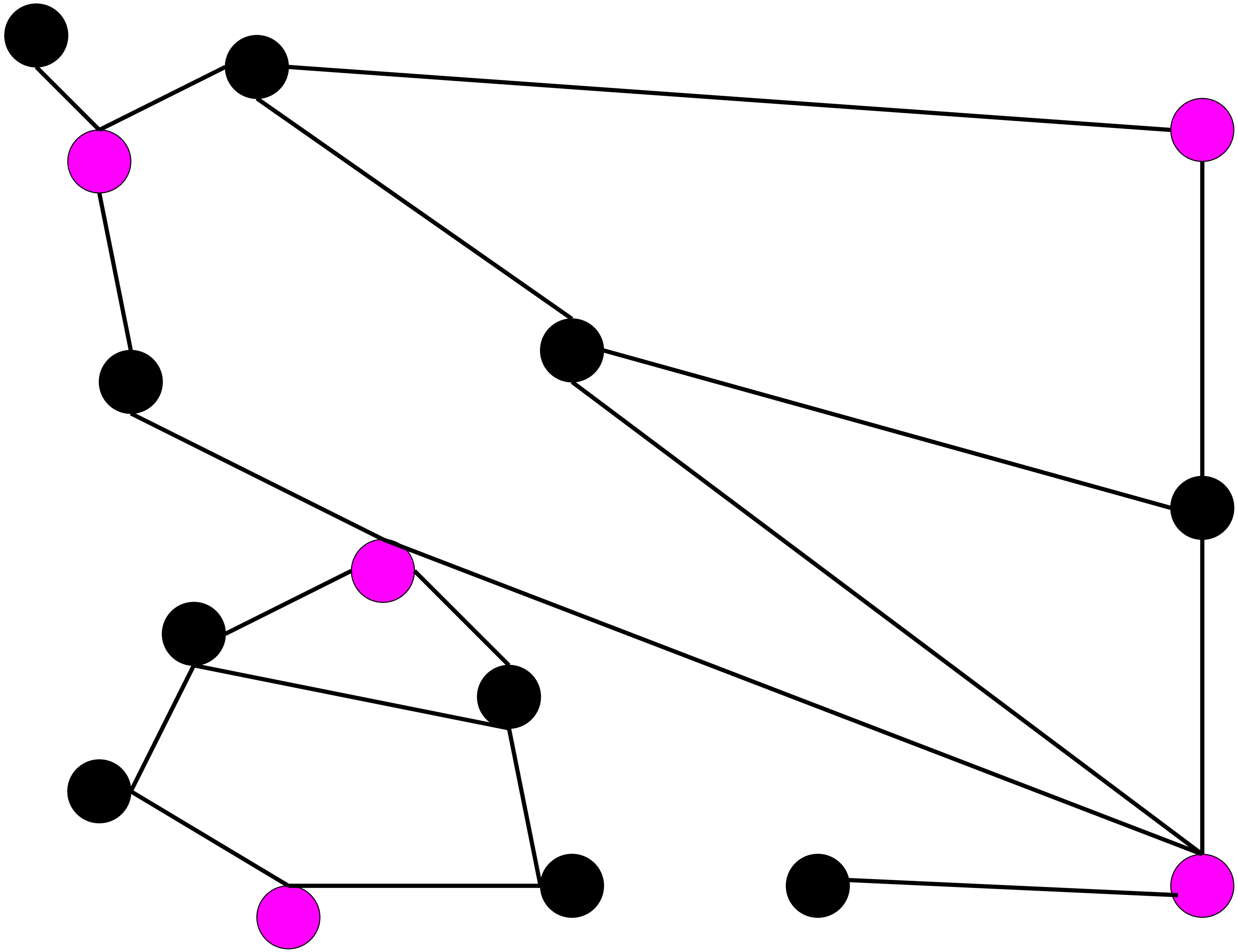








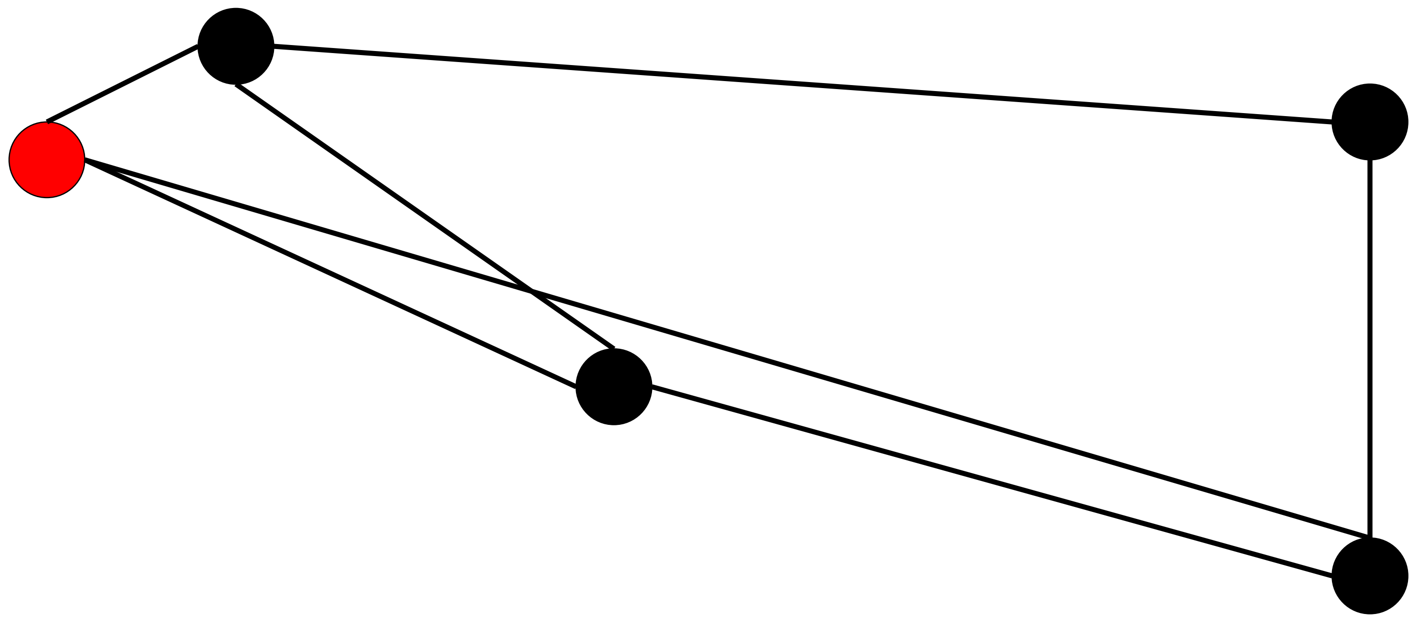




A formal problem:

The reduced instance contains the catalyst.

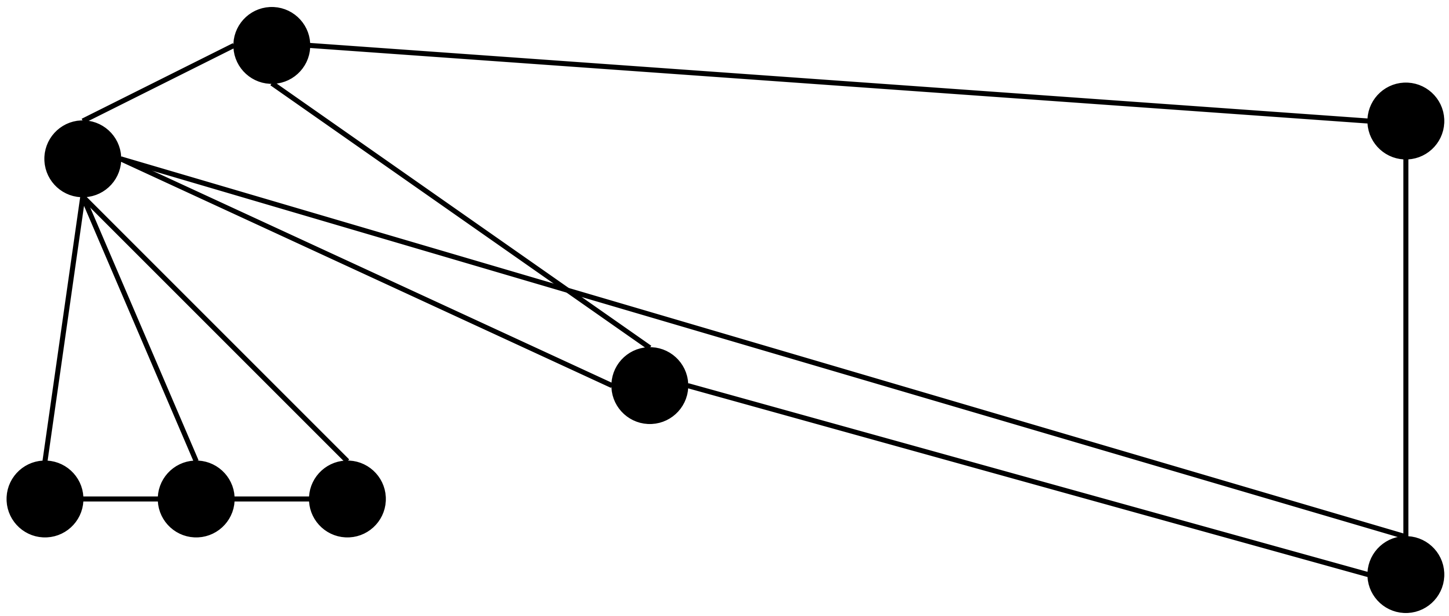
$\Rightarrow$ This is NOT a NONBLOCKER SET instance.



How to get rid of the catalyst

Reduction rule 7 (De-catalyzation rule) *Let* (G, c, k_d) *be an instance of* NONBLOCKER SET WITH CATALYTIC VERTEX*. Then, perform the following surgery to obtain a new instance* (G', k'_d) *of* NONBLOCKER SET *(i.e., without a catalytic vertex):*

Add three new vertices u*,* v*, and* w *and introduce new edges* cu*,* cv*,* cw*,* uv *and* vw*. All other vertices and edge relations in* G *stay the same. This describes the new graph* G'*. Set* $k'_d = k_d + 3$*.*



The overall kernelization algorithm

1. Apply catalyzation rule.

2. Exhaustively apply all reduction rules (besides decat. and cat.).

3. Apply de-catalyzation rule.

4. If reduced graph has at least $(5/3)k_d + 3$ vertices, YES.

5. Otherwise: We have a graph instance with less than $(5/3)k_d + 3$ vertices.

More algorithmic consequences

. . . see proceedings

Take Away

- Knowing good mathematical theorems helps a lot in kernelization (and search tree) algorithmics.

- Conversely, parameterized algorithmics might stir interest in and motivate developing structural (e.g., graph-theoretic) results.