A General Data Reduction Scheme for Domination in Graphs

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Outline

Dominating Set

1. Definitions
2. Reduction rules
3. Generalization of the rules
4. Hierarchy, reduction scheme

Outline

II Directed Dominating Set

1. Definition
2. Reduction rules
3. Directed Dominating Set on planar graphs

Dominating Set

given: $G = (V, E)$, undirected graph
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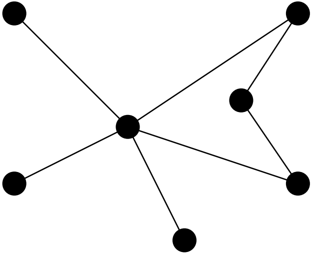

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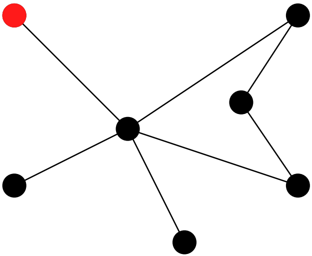

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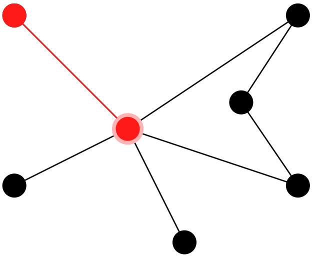

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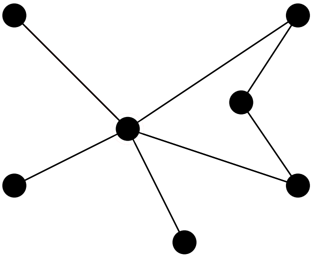

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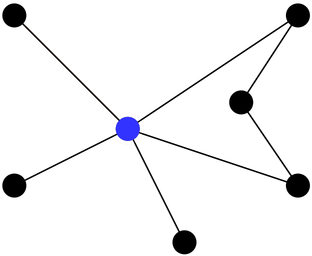

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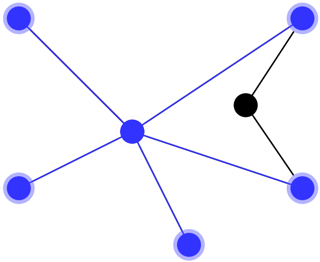

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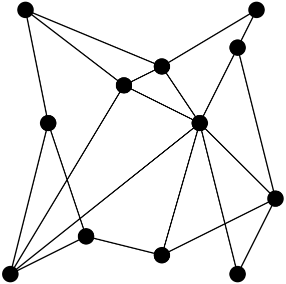

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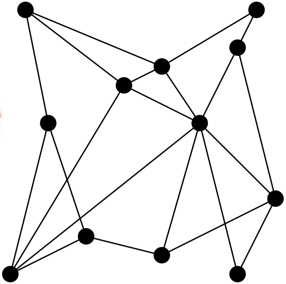

Fire station problem

For security reasons, each city needs a fire station

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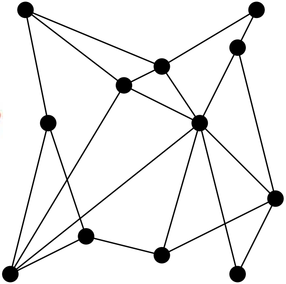

Fire station problem

For security reasons, each city needs a fire station — at least one in a city of its neighborhood

Dominating Set

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- NP-complete

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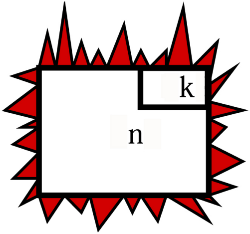
- NP-complete
- fixed-parameter tractable with parameter k on planar graphs

Fixed-parameter tractability

Idea: Restrict the seemingly inherent combinatorial explosion of hard problems to some problem-specific parameters.



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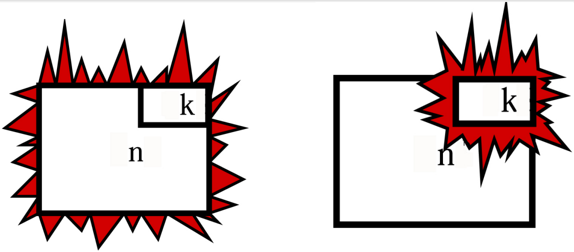

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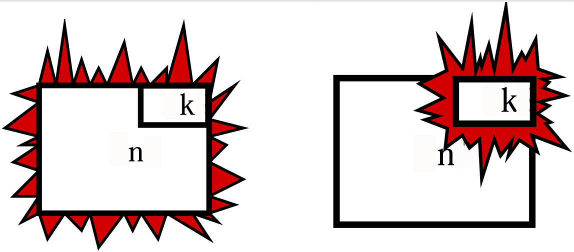

Definition (fixed-parameter tractable)

Problem P is fixed-parameter tractable

$$\Longleftrightarrow$$

P is solvable in $O(f(k) \cdot n^c)$ time

Dominating Set

Dominating Set is

- NP-complete
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Dominating Set

Dominating Set is

- NP-complete
- fixed-parameter tractable with parameter k on planar graphs
- W[2]-complete

Data reduction

Data reduction

Idea (for DOMINATING SET:)

Replace (in polynomial time) a given instance (G, k)
by "simpler" instance (G', k')

such that

(G, k) is a yes-instance $\iff$ (G', k') is a yes-instance

Reduction rules

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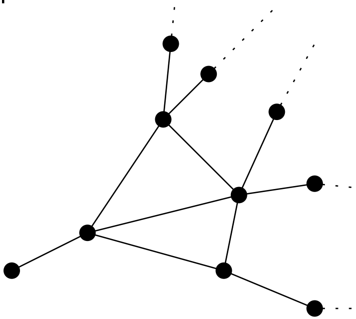

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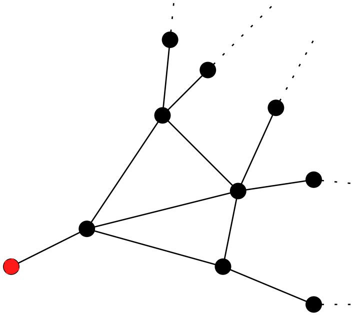

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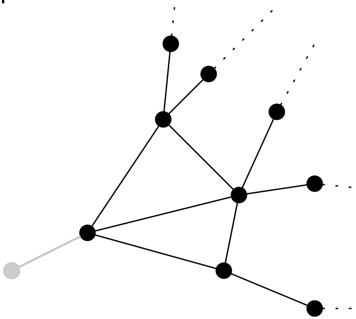
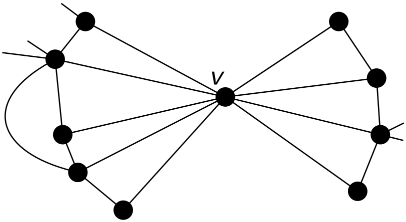


Reduction rules

closed Neighborhood $N[v]$ of a vertex $v \in V$



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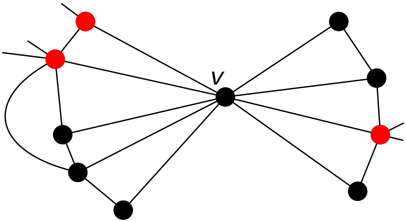

$N_{\text{exit}}(v)$: vertices possessing neighbors outside of $N[v]$

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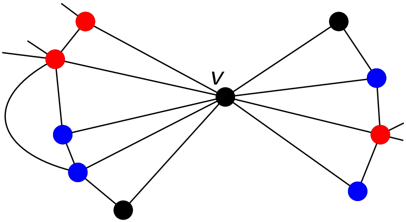

$N_{\text{exit}}(v)$: vertices possessing neighbors outside of $N[v]$
$N_{\text{guard}}(v)$: neighbors of exit-vertices in $N(v)$

Reduction rules

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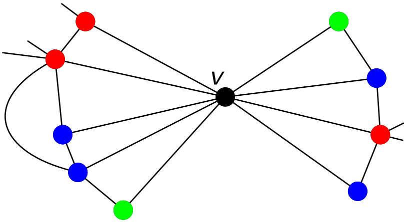

$N_{exit}(v)$: vertices possessing neighbors outside of $N[v]$
$N_{guard}(v)$: neighbors of exit-vertices in $N(v)$
$N_{prison}(v)$: remaining vertices in $N(v)$

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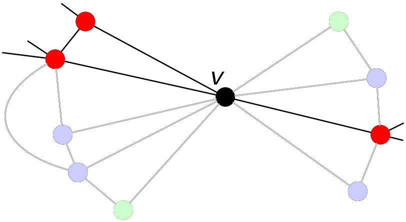
closed Neighborhood $N[v]$ of a vertex $v \in V$



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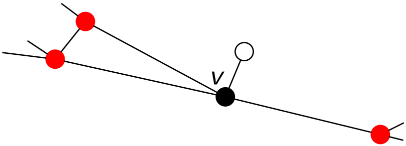


Reduction rules

Rule 1

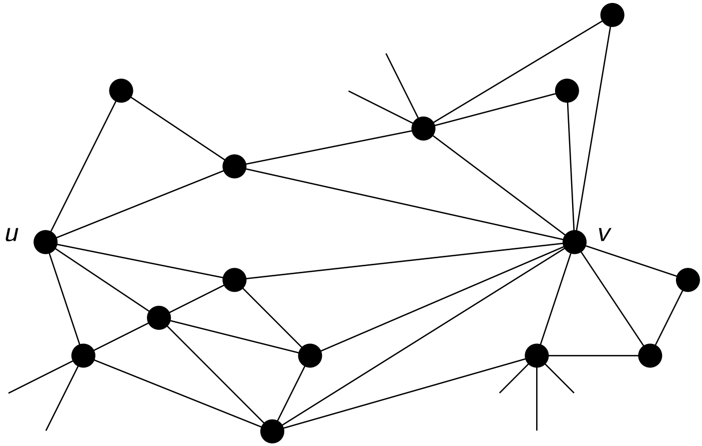
if $N_{\text{prison}}(v) \neq \emptyset$

then

- choose v to belong to the dominating set (add "gadget vertex")
- remove $N_{\text{guard}}(v) \cup N_{\text{prison}}(v)$

Reduction rules

Neighborhood of 2 vertices $u, v \in V$



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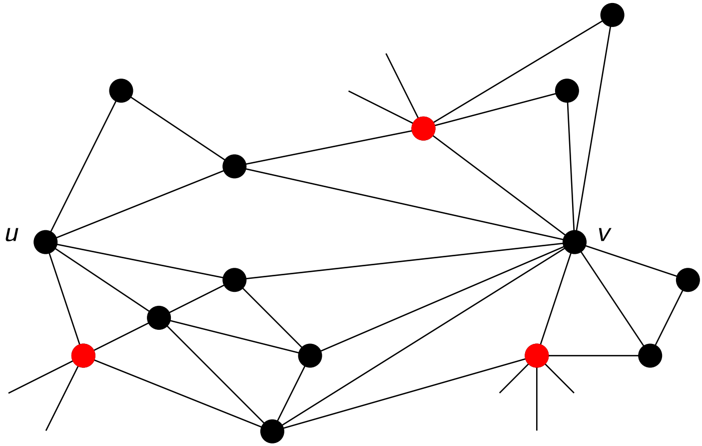

$N_{\text{exit}}(u, v)$

Reduction rules

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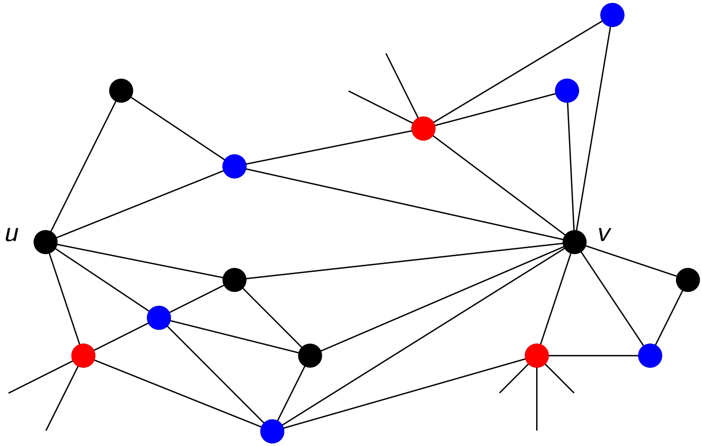
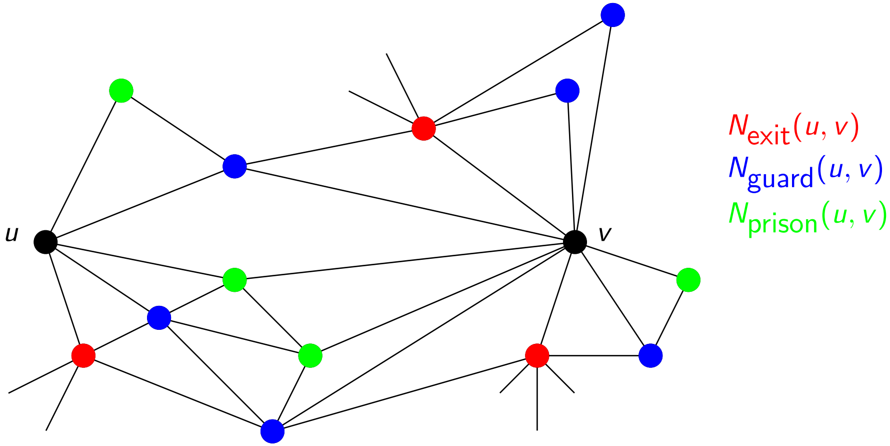


$N_{\text{exit}}(u, v)$
$N_{\text{guard}}(u, v)$

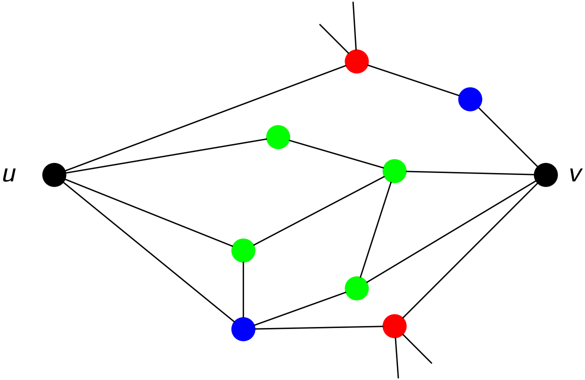
Reduction rules

Neighborhood of 2 vertices $u, v \in V$



Reduction rules

Problem: all prison-vertices might be dominated by one single prison-vertex



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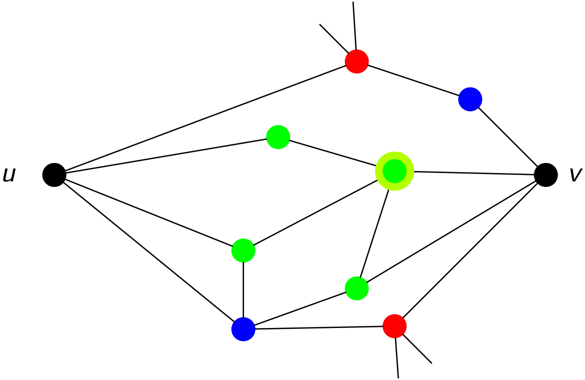

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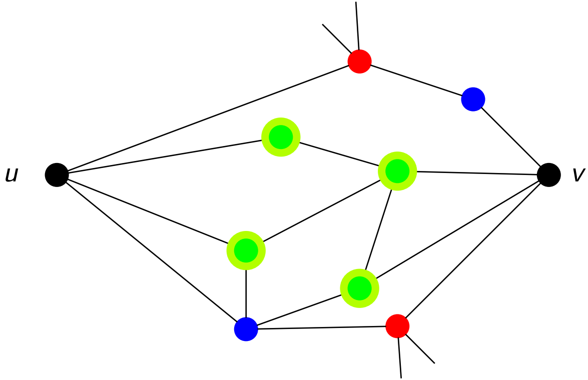

One single prisoner-vertex could be better
$\Rightarrow$ don't apply rule

Reduction rules

Rule 2

if $N_{\text{prison}}(u, v) \neq \emptyset$

and $N_{\text{prison}}(u, v)$ cannot be dominated
by a single guard- or prison-vertex

then:

Reduction rules

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if $N_{\text{prison}}(u, v) \neq \emptyset$

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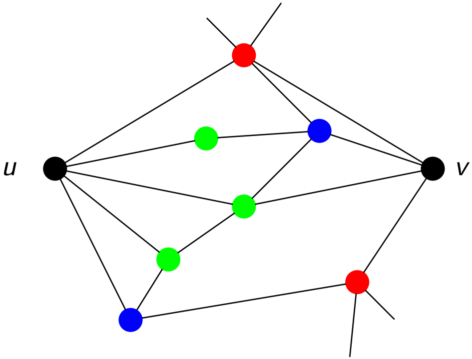
then:
distinguish 3 cases

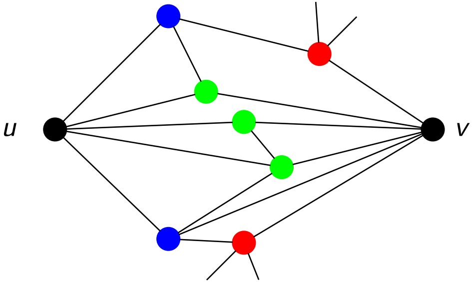
Reduction rules



Rule 2

case 1

Reduction rules



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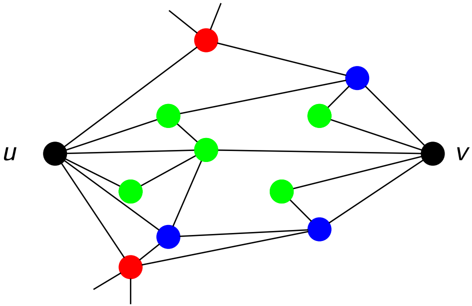

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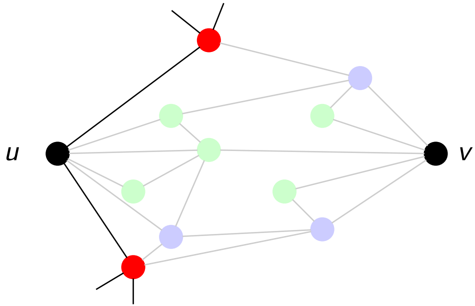

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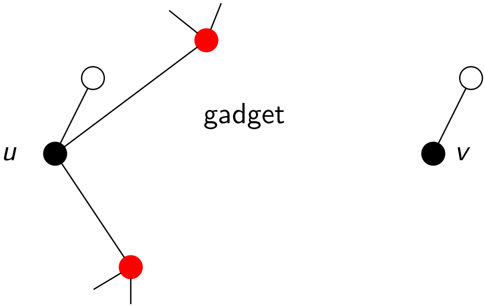
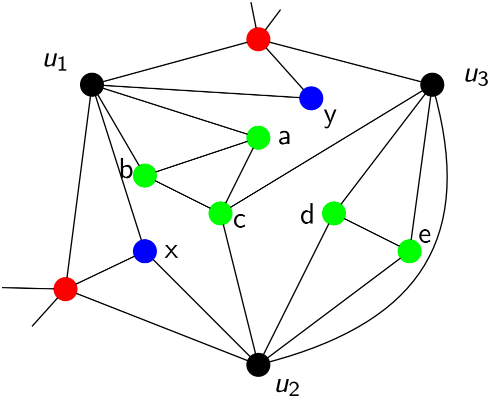


Rule 2

case 3

generalizing the rules for r vertices



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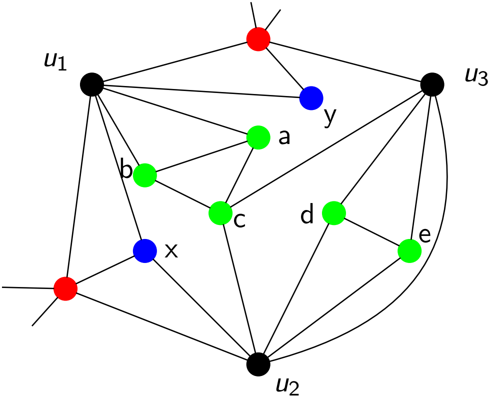

subsets of $\{u_1, u_2, u_3\}$

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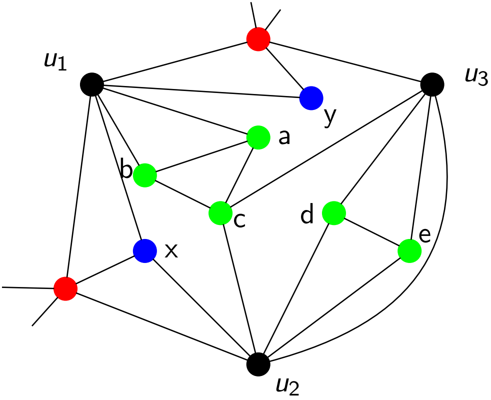

subsets of $\{u_1, u_2, u_3\}$

$W =$
$\{\{u_1, u_2\}, \{u_1, u_3\}, \{u_1, u_2, u_3\}\}$

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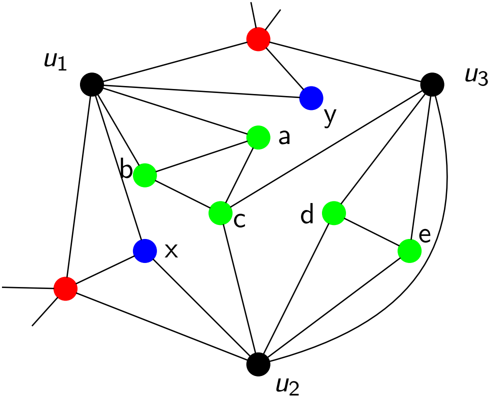

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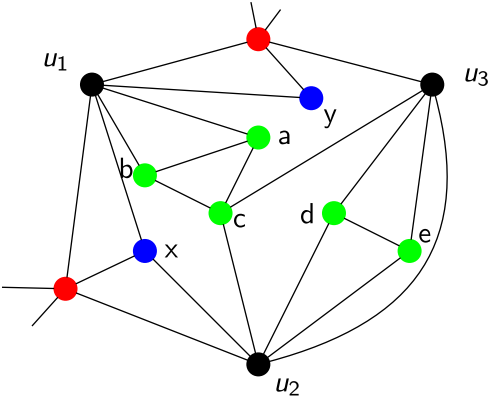

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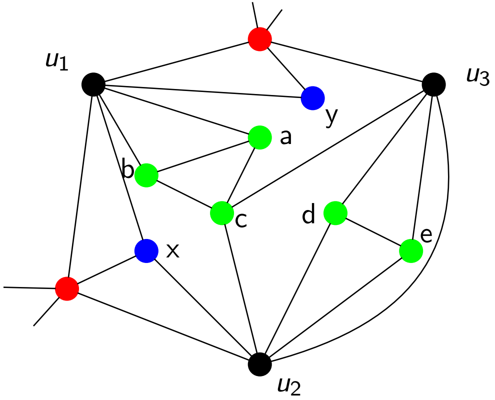
alternatives

generalizing the rules for r vertices



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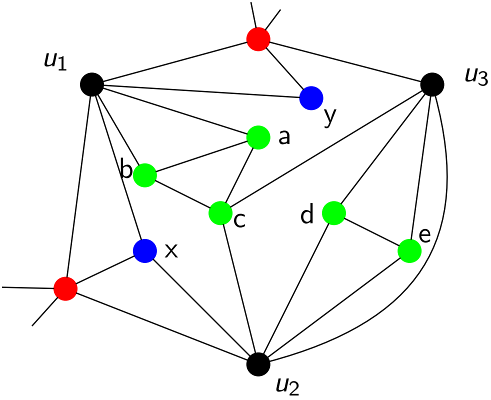
alternatives

$W_{\text{alt}} =$
$W \cup \{\{a, d\}, \{b, d\}, \{c, d\},$
$\{x, a, d\}, \{u_2, a\}, \dots\}$

generalizing the rules for r vertices



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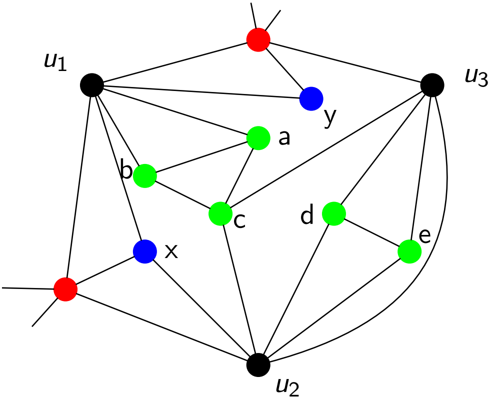
$\{u_2, a\}, \{u_2, b\}, \{u_3, a\}, \{u_3, b\}\}$

generalizing the rules for r vertices



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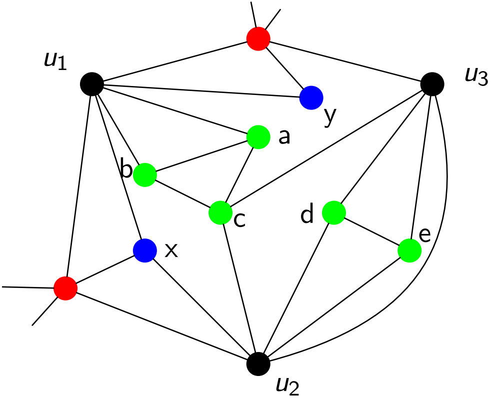

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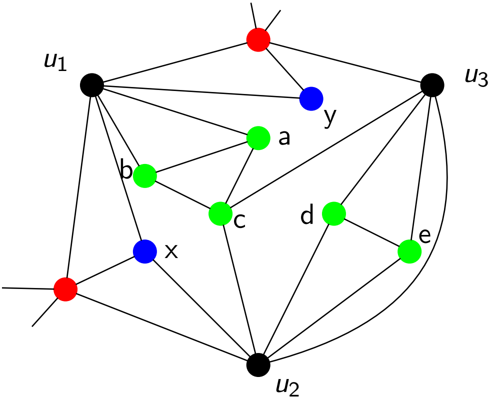

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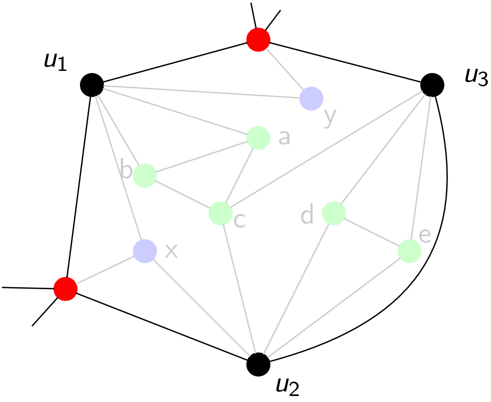
Rule r

if $\widehat{W}$ is “better” than $\widehat{W}_{\text{alt}}$
then

generalizing the rules for r vertices



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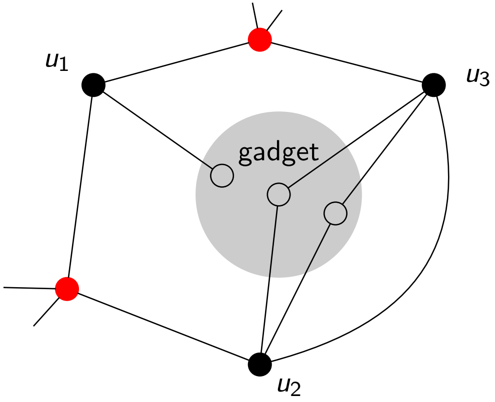
$\widehat{W}_{\text{alt}} = \{\{u_1, u_2\}, \{u_1, u_3\}, \{a, d\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}, \{u_2, a\}, \{u_2, b\}, \{u_3, a\}, \{u_3, b\}\}$

Rule r

if $\widehat{W}$ is "better" than $\widehat{W}_{\text{alt}}$ then

- remove all vertices in $N_{\text{prison}} \cup N_{\text{guard}}$ which are neighbors of *all* elements of $\widehat{W}$

generalizing the rules for r vertices



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if $\widehat{W}$ is "better" than $\widehat{W}_{\text{alt}}$ then

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- insert a gadget

Do we really need the r-Rules?

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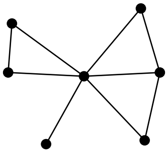
Definition:

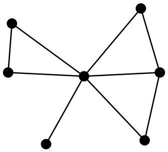
A graph $G = (V, E)$ is said to be *reduced with respect to r-Rule* if there is no set of distinct vertices $u_1, \ldots, u_r$ for which r-Rule can be applied.

Do we really need the r-Rules?



Do we really need the r-Rules?

not reduced



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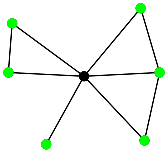

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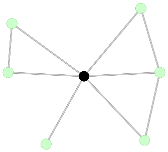

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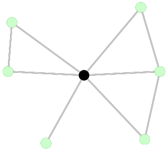

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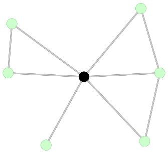
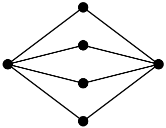


graphs reduced with respect to 1-Rule



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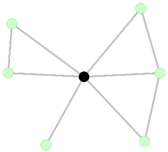

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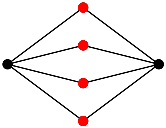

Do we really need the r-Rules?

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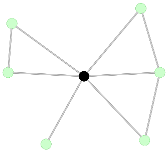
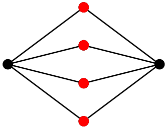
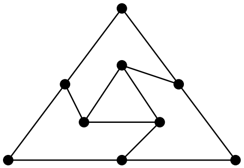
graphs reduced with respect to

1-Rule

2-Rule







Do we really need the r-Rules?

graph reduced with respect to 1-Rule, 2-Rule, . . . , $(r-1)$-Rule:

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grid of width 2, length $2r-2$

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Theorem

The reduction scheme given by r-Rule builds a strict hierarchy of rules

Running time

$G = (V, E), \quad |V| = n$

1-Rule: $\mathcal{O}(n^3)$

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r-Rule: $\mathcal{O}(n^{2r})$

Directed Dominating Set

given: $\vec{G} = (V, \vec{E})$, directed graph
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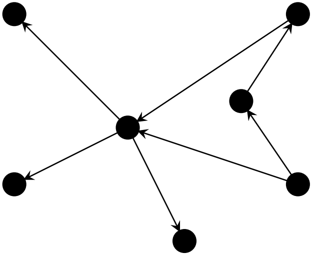

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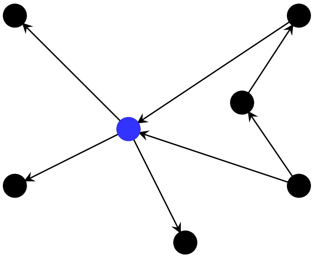

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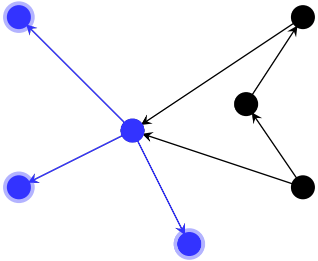

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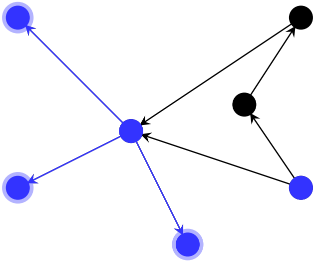

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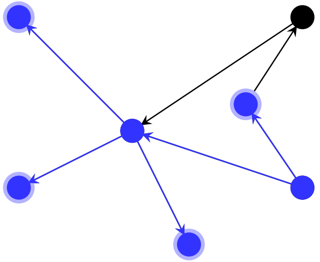
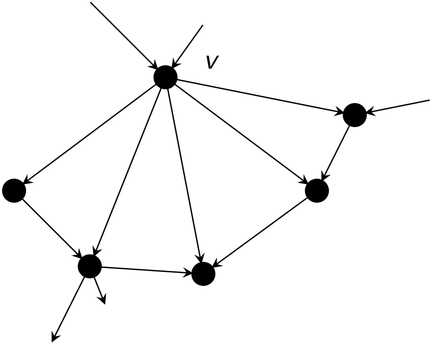


Directed Dominating Set — reduction rules

Neighborhood of one vertex $v \in V$



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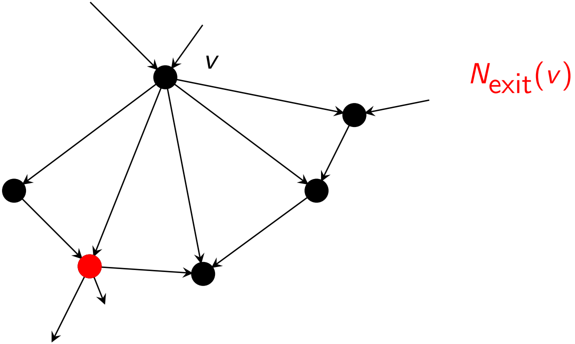

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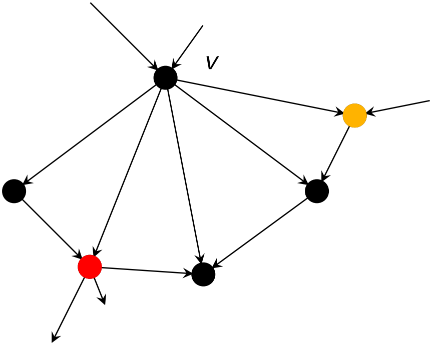

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$N_{\text{enter}}(v)$

Directed Dominating Set — reduction rules

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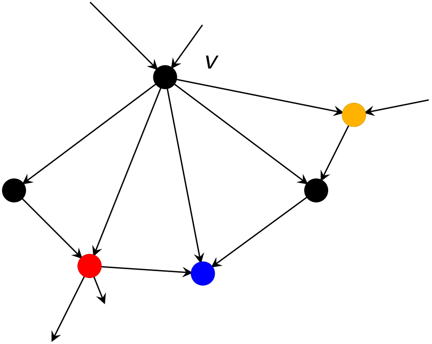

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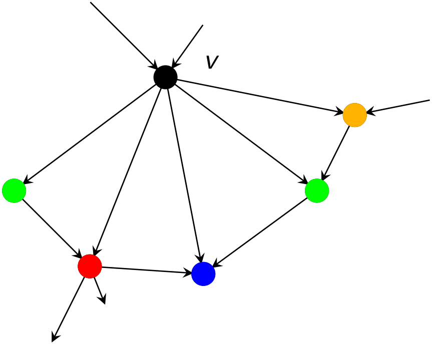

$N_{exit}(v)$

$N_{enter}(v)$

$N_{guard}(v)$

$N_{prison}(v)$

Reduction rules for Directed Dominating Set

starting rule:

$\vec{0}$-Rule:

Reduction rules for DIRECTED DOMINATING SET

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Reduction rules for DIRECTED DOMINATING SET

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- choose v to belong to the dominating set

Reduction rules for DIRECTED DOMINATING SET

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Reduction rules for DIRECTED DOMINATING SET

$\vec{1}$-Rule:

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$\vec{1}$-Rule:

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Reduction rules for Directed Dominating Set

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$\vec{r}$-Rule:

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Final result for Directed Dominating Set

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Theorem

On planar graphs, Directed Dominating Set is fixed-parameter tractable.

Conclusion

- Dominating Set
  - Generalization of reduction rules 1, 2 to r

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Open questions

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