SOFSEM07

The P_k partition problem and related problems in bipartite graphs

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Outline

1 - Introduction

- $\mathbf{P}_k$ and k-Path *partition* and *packing* problems
- Complexity
- Approximation

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- A reduction from k-DM to $\mathbf{P}_k$PACKING
- **NP**-completeness
- **APX**-hardness

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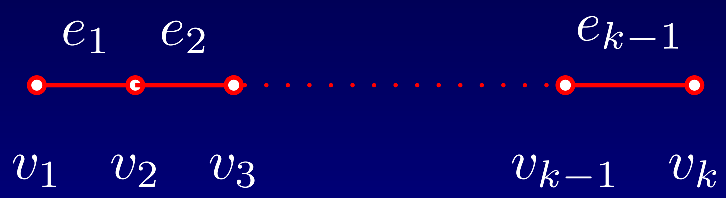

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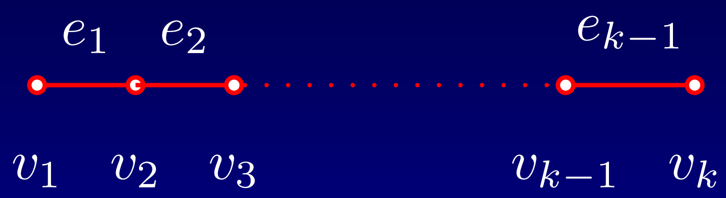

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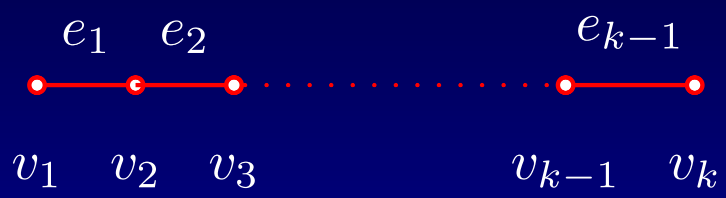

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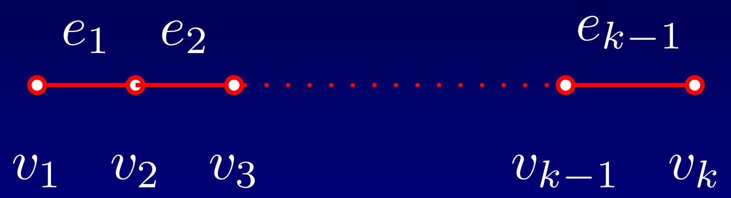

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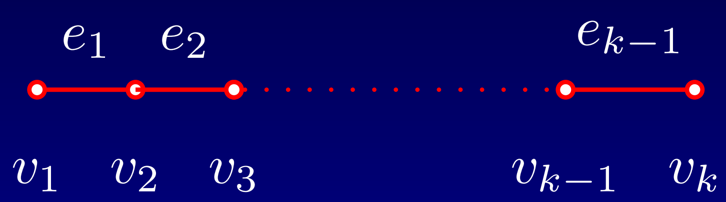

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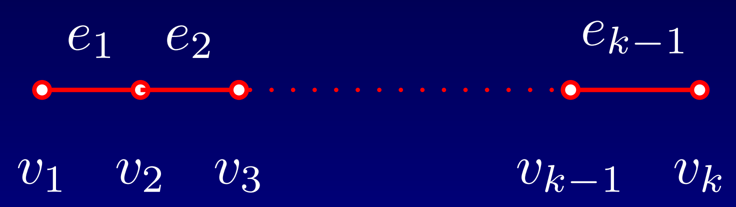

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Relations

- $\mathbf{P}_k$PARTITION $\propto$ $\begin{cases} \text{MAX(W)}\mathbf{P}_k\text{PACKING, MAXW}\mathbf{P}_k\text{PARTITION,} \\ \text{MIN}k\text{-PATHPARTITION} \end{cases}$

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- For any value of k $\left\{\begin{array}{l} \text{maximum degree 2} \\ \text{forest} \end{array}\right\} \Rightarrow$ all are **P**

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- MAX(W)$\mathbf{P}_k$PACKING
  $\longrightarrow 1/(k-1)-\varepsilon\ \forall k$ (as a special max weighted packing problem, [1])
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- MAX$\mathbf{P}_3$PACKING and MAXW$\mathbf{P}_3$PACKING: 1/2 on
  - bipartite (MAX$\mathbf{P}_3$PACKING)
  - bipartite + maximum degree 3 (MAXW$\mathbf{P}_3$PACKING)

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- MIN3-PATHPARTITION 3/2-approximable

2. k DM $\propto$ P_kPACKING

kDIMENTIONAL MATCHING

- Instance

  $I = (X_1 \times \ldots \times X_k, \mathcal{C})$

  $|X_\ell| = n \,\forall \ell, \quad X_\ell \cap X_h = \emptyset \,\forall \ell \neq h$

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Claim

$\forall k \geq 3$, kDM maps to $\mathbf{P}_k$PACKING in bipartite graphs with max degree 3

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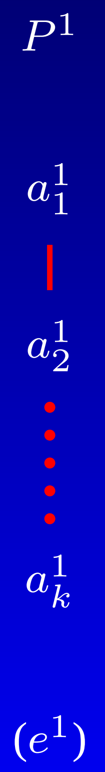

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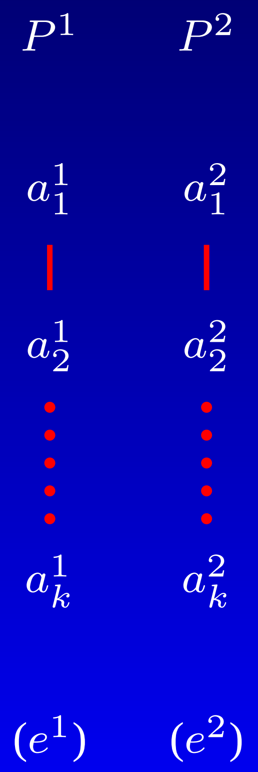

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P^1	P^2	$\cdots\cdots\cdots\cdots$	P^k
a_1^1	a_1^2		a_1^k
a_2^1	a_2^2		a_2^k
$\vdots$	$\vdots$		$\vdots$
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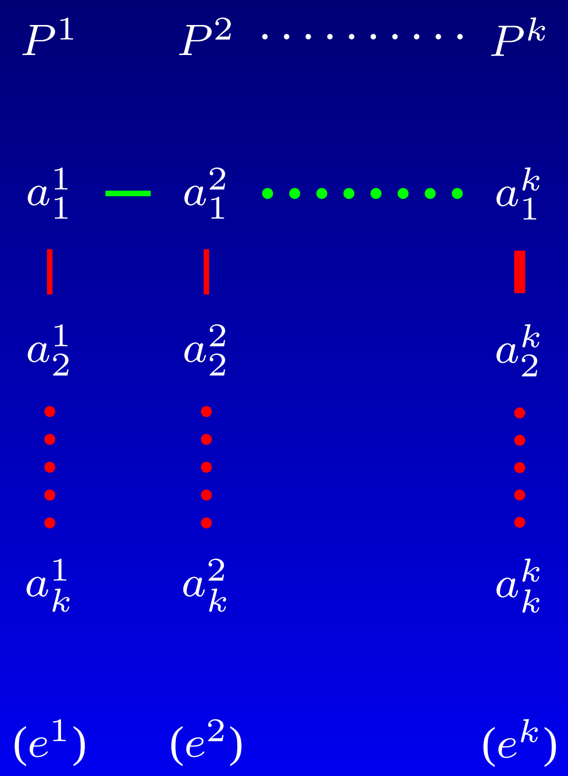

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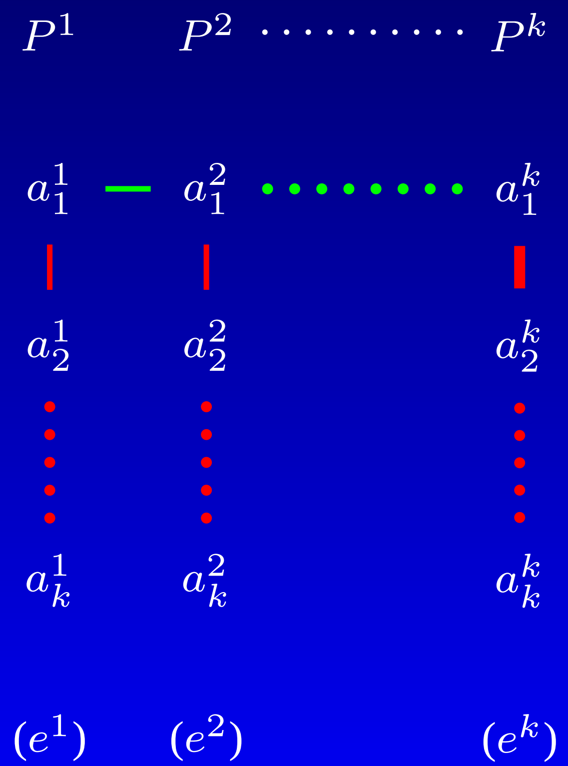

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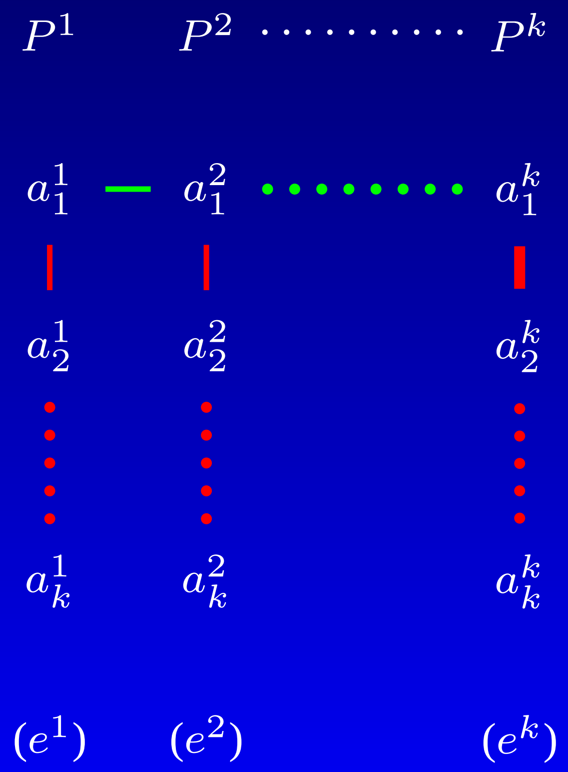

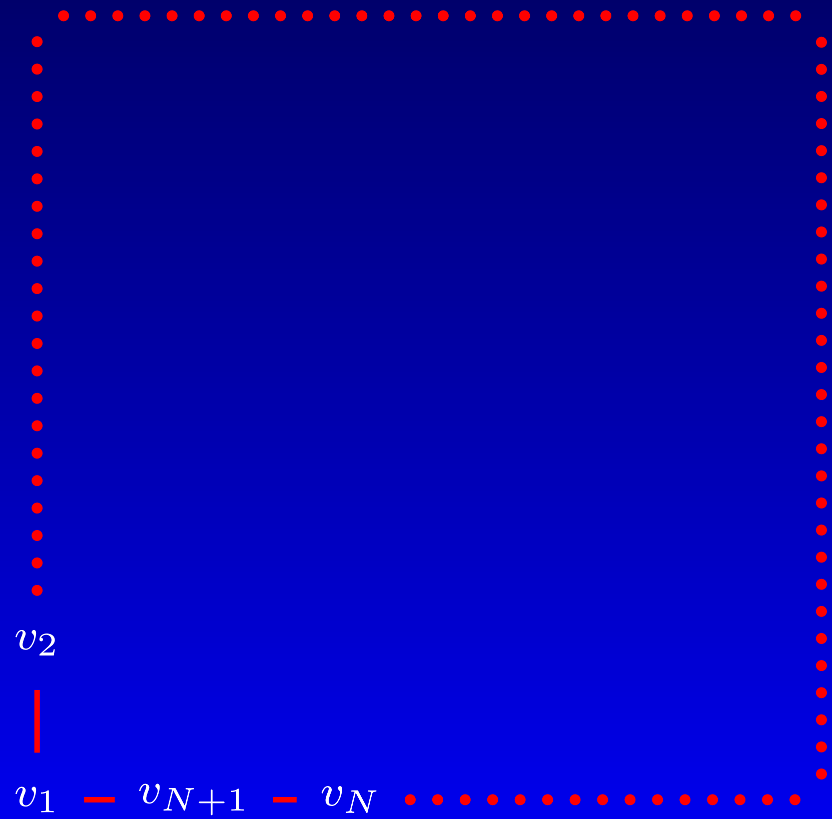

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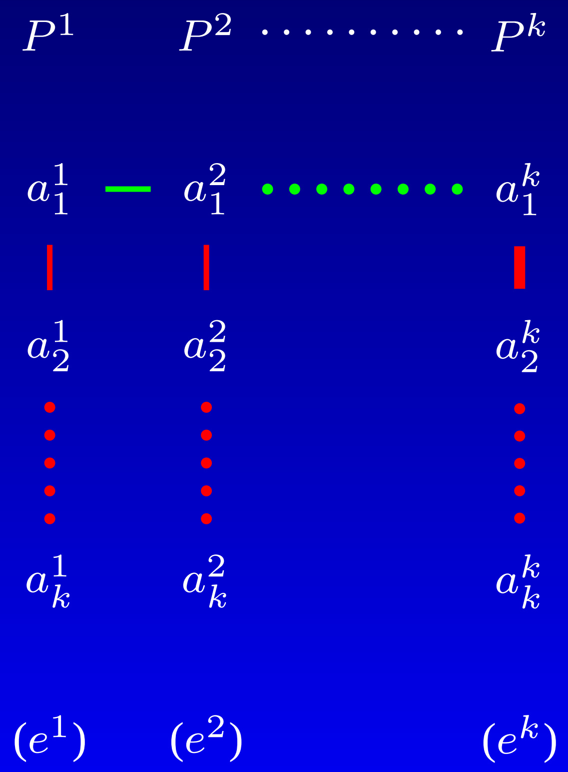

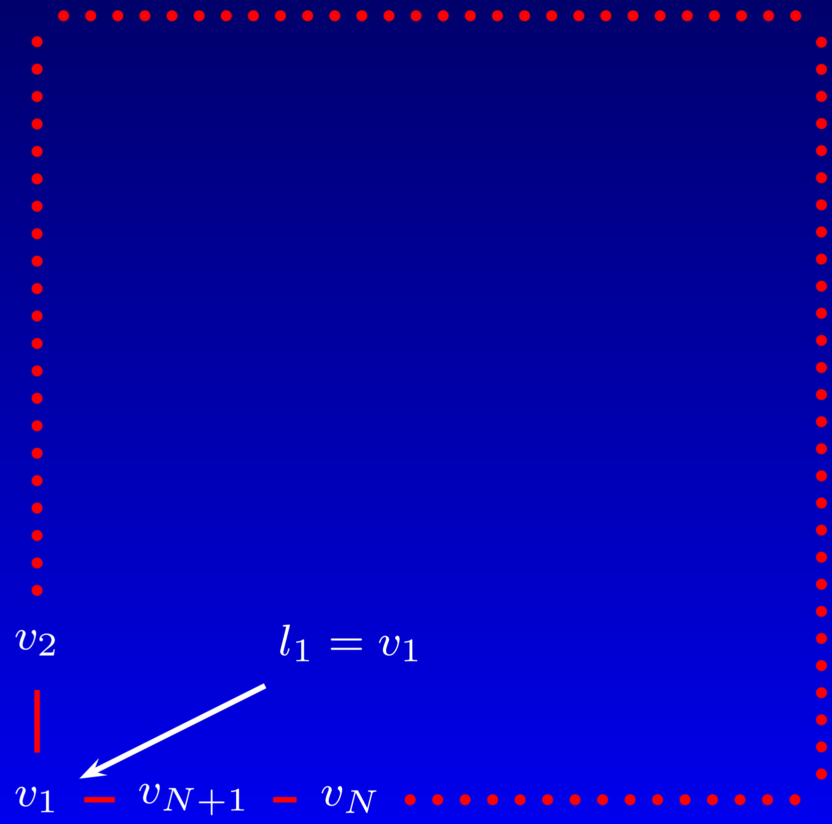

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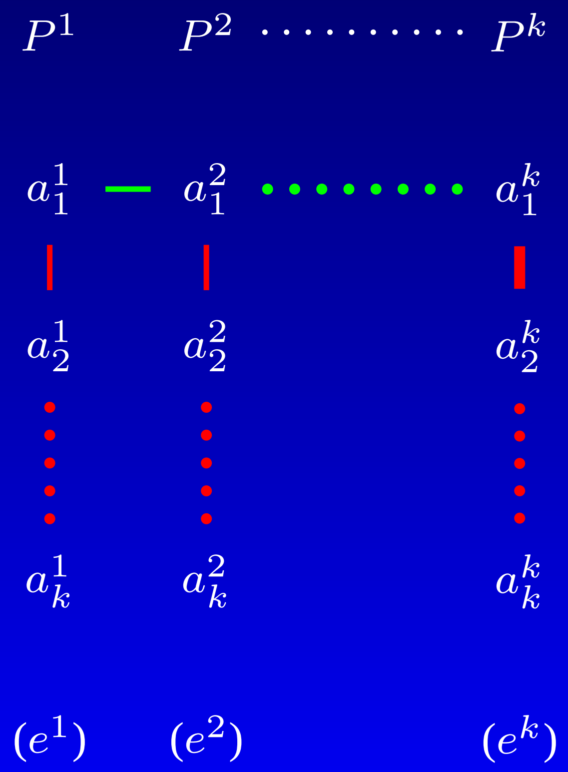

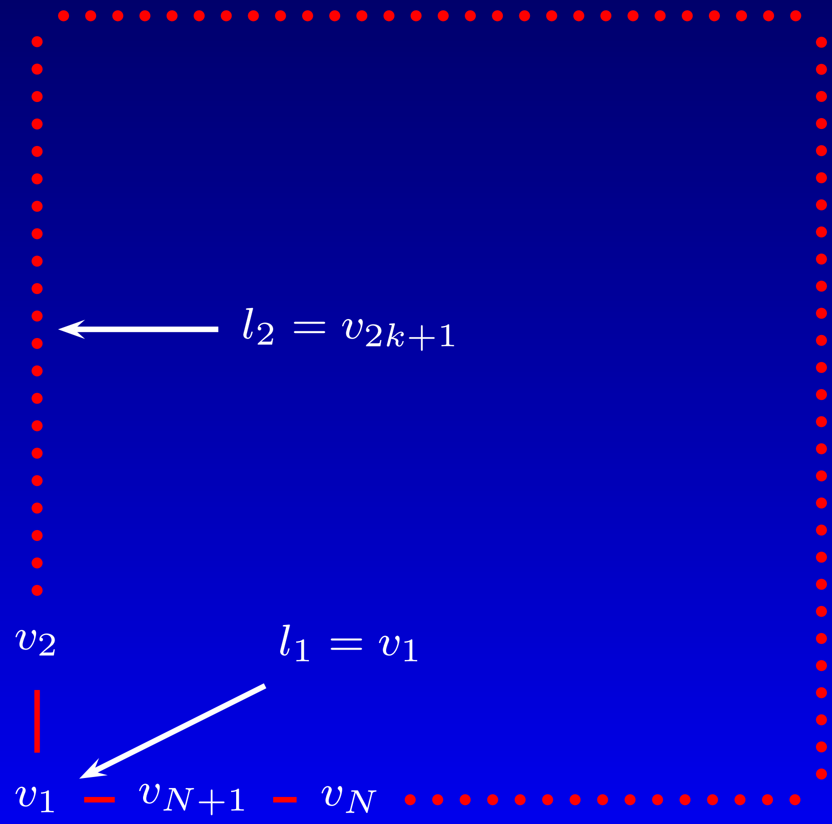

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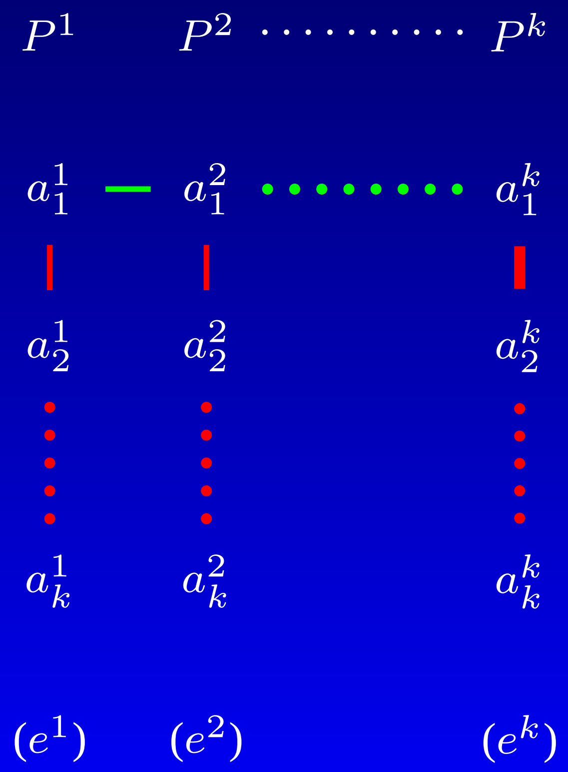

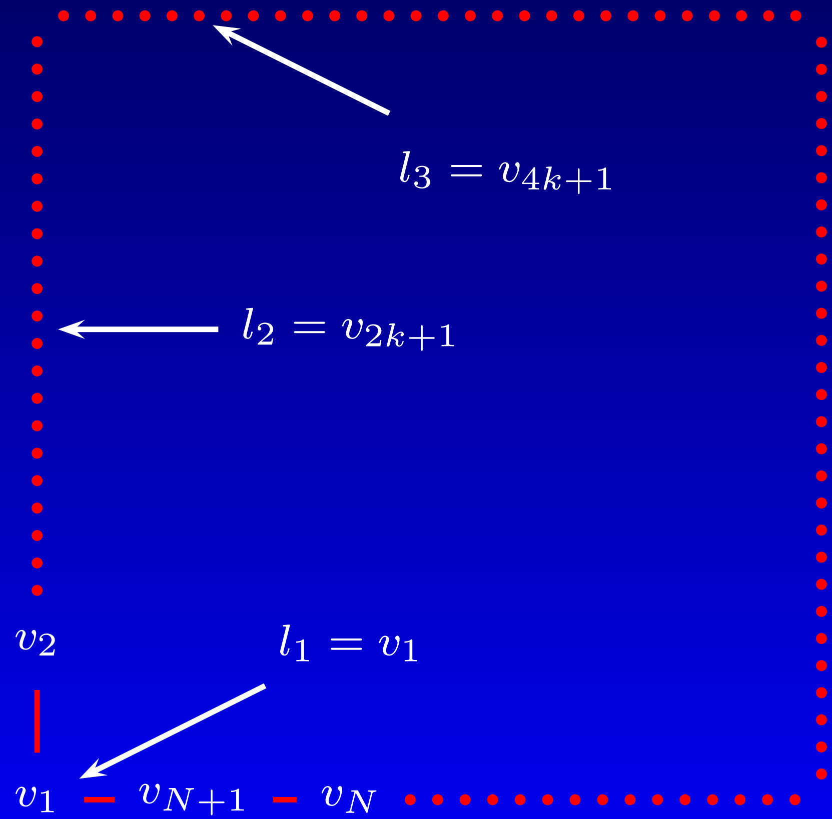

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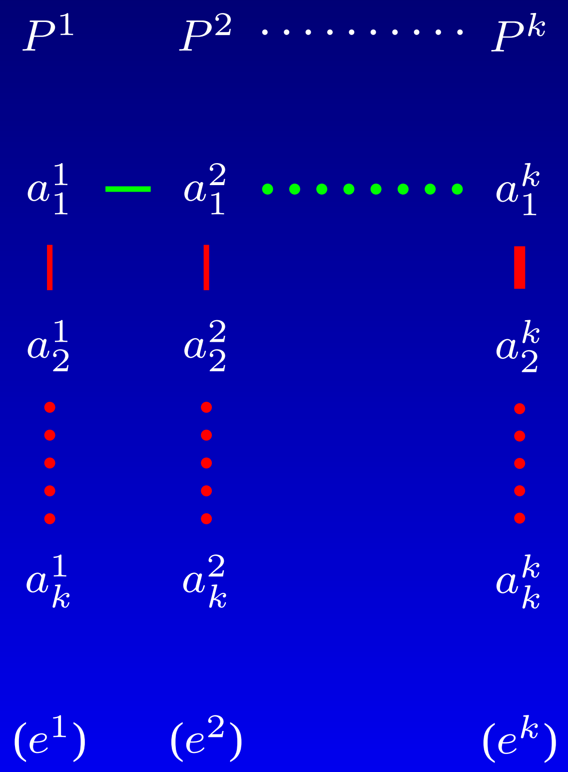

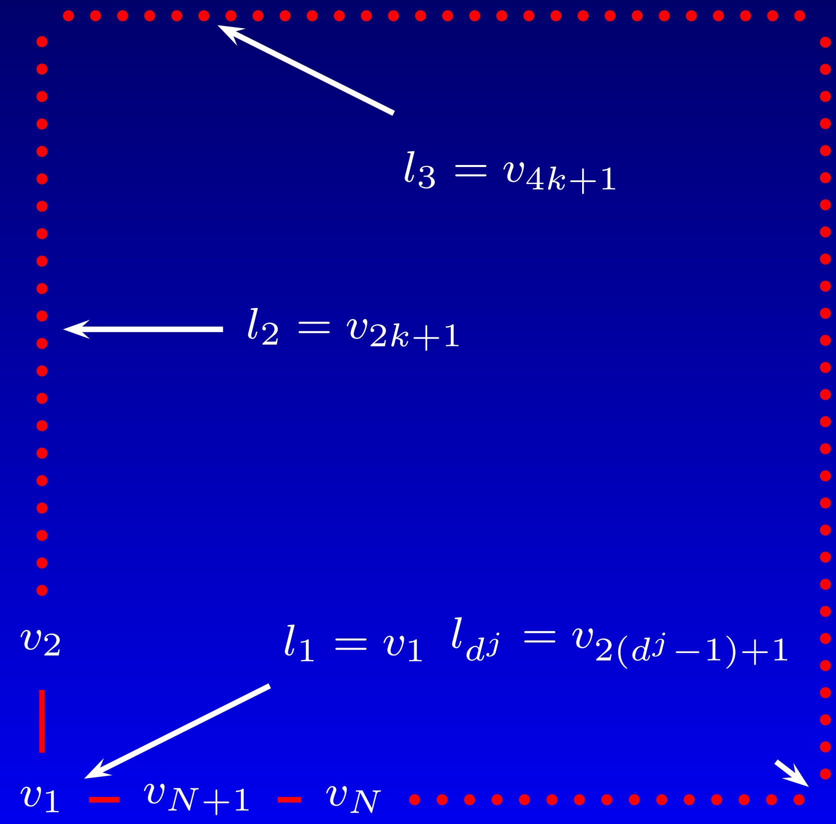

2. Proof (k odd)

$$I = (X, \mathcal{C}) \text{ instance of } k\text{DM} \mapsto G = (V, E)$$

$$c_i = (e^1, e^2, \ldots, e^k) \in \mathcal{C} \mapsto H(c_i)$$

$$e_j \in X_1 \cup \ldots \cup X_k \mapsto H(e_j)$$

$$N = k \times (2d^j - 1) \quad d^j = d(e_j)$$

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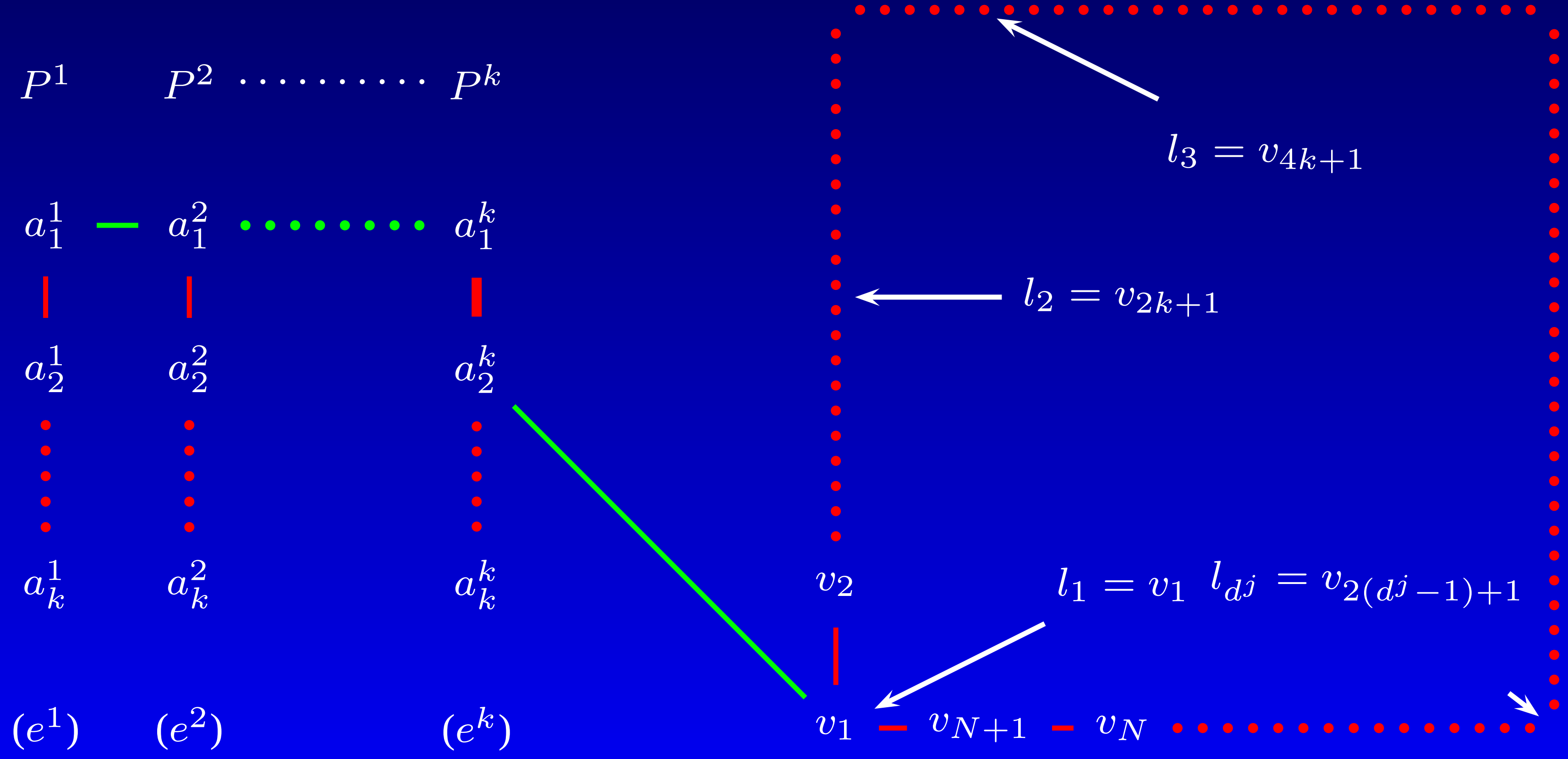

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Properties on G

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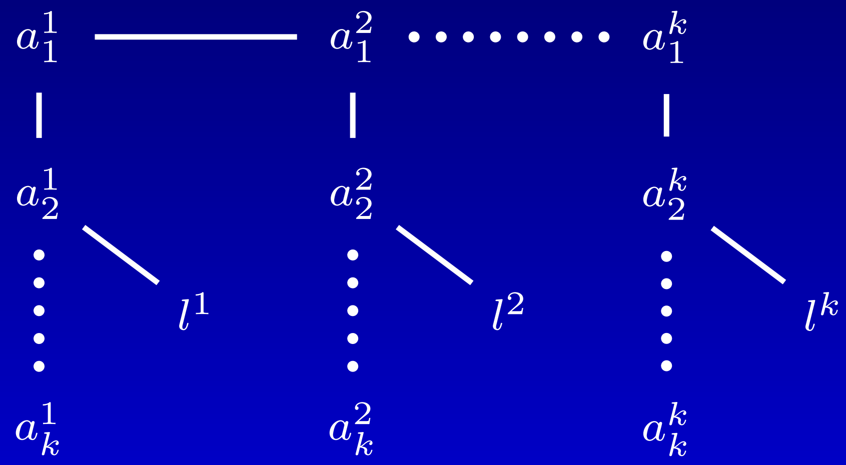

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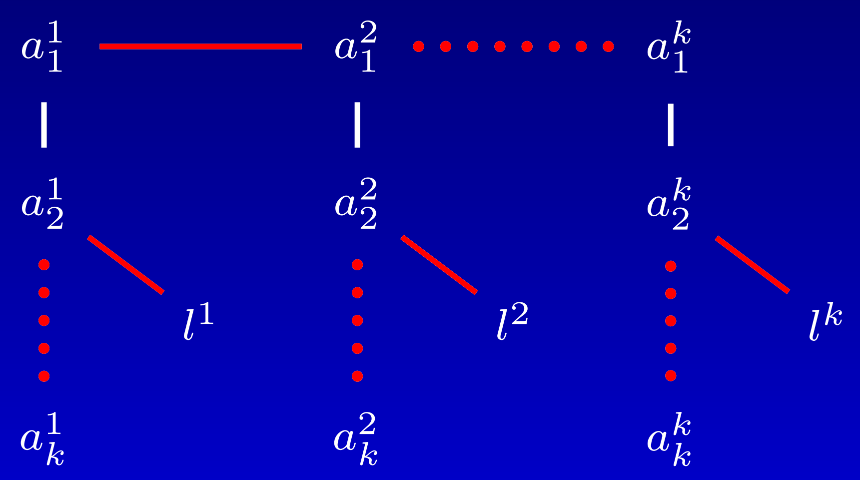
$|\mathcal{P}^i| = k + 1$



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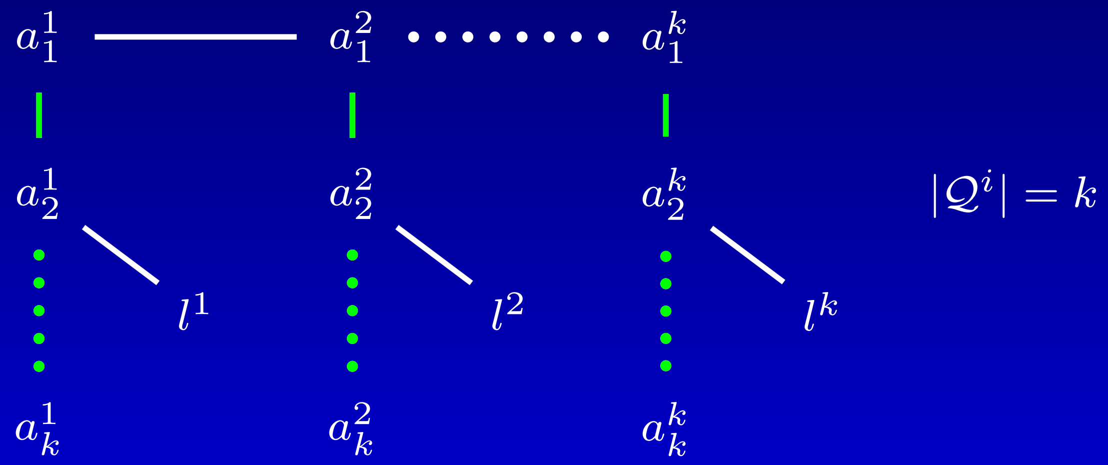

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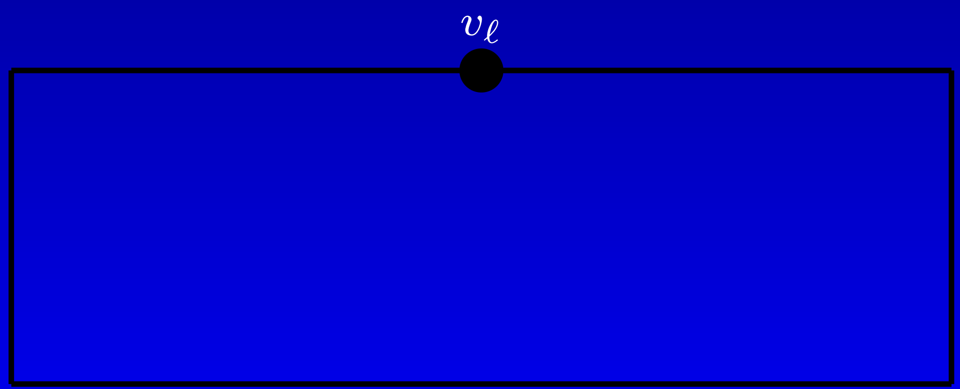

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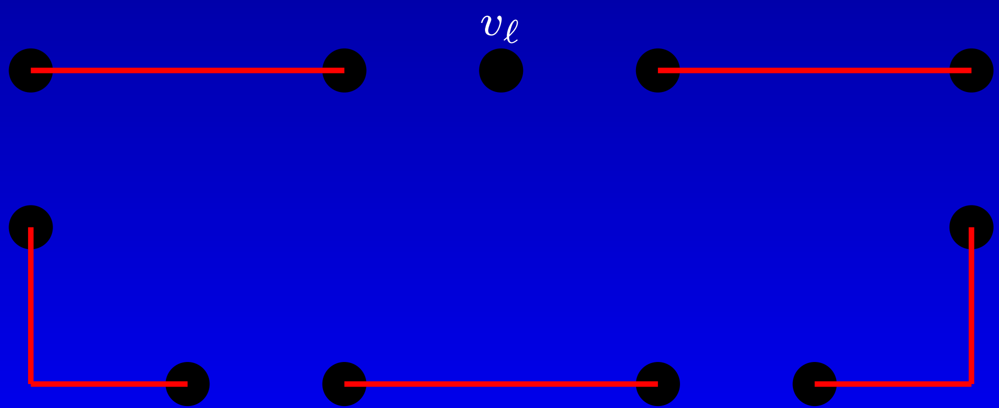

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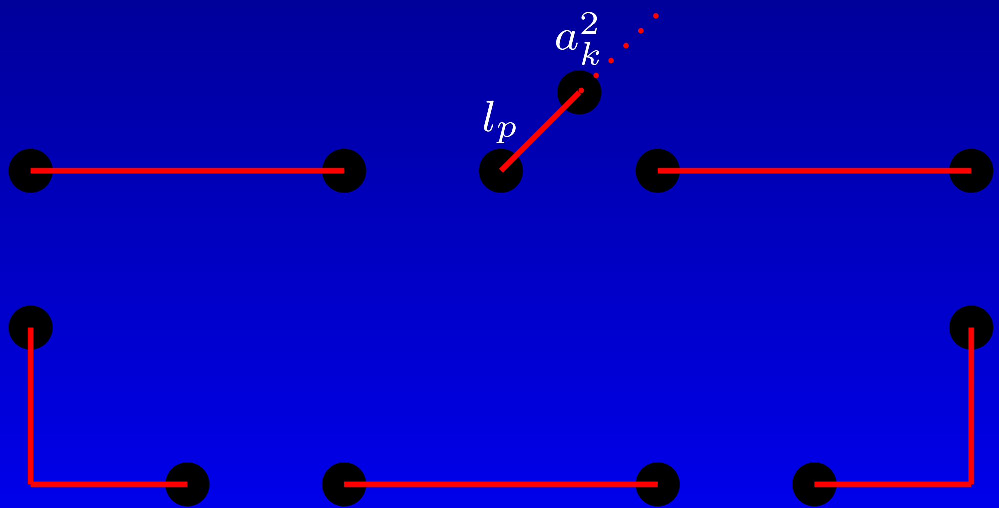

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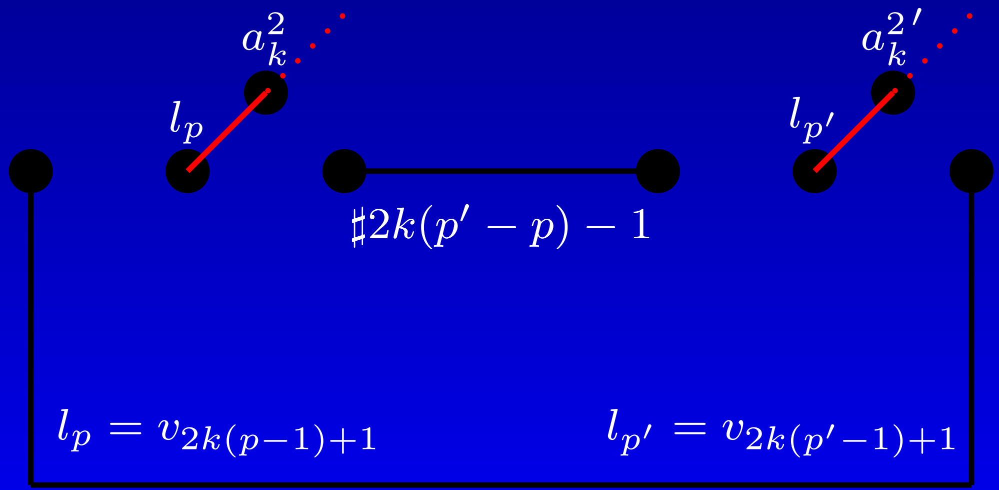

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  $\Rightarrow |\mathcal{P}'| = (km + \mu) + (2km - kn) = \mu + (3km - kn)$

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Relations between I and G

- M matching on $I \longrightarrow \mathcal{P}'$ packing on G s.t. $|\mathcal{P}'| = |M| + (3km - kn)$

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- $\Rightarrow\ |\mathcal{P}^*| = |M^*| + (3km - kn)$

2. k DM $\propto$ P_kPACKING

$$|\mathcal{P}^*| = |M^*| + (3km - kn) \qquad |V| = k\,(n + (3km - kn))$$

Consequences

- $\exists\, M$ perfect matching on $I \;\Leftrightarrow\; \exists\, \mathcal{P}'\ \mathbf{P}_k$ partition on G
  $\Rightarrow\ \mathbf{P}_k$PARTITION is $\mathbf{NP} - \mathbf{c}$ in bipartite max deg 3 $\forall k$
  $\Rightarrow\ \mathbf{P}_3$PARTITION is $\mathbf{NP} - \mathbf{c}$ in planar bipartite max deg 3

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  $\Rightarrow$ $\mathbf{P}_3$PARTITION is $\mathbf{NP}-\mathbf{c}$ in planar bipartite max deg 3
- $|M^*| = |\mathcal{P}^*| - (3km - kn)$ $\Rightarrow$

$$\begin{cases} |M^*| = n \\ |M^*| < (1-\varepsilon)n \end{cases} \Leftrightarrow \begin{cases} |\mathcal{P}^*| = n + (3km - kn) \\ |\mathcal{P}^*| < (1-\varepsilon')\,(n + (3km - kn)) \end{cases}$$

whith $\varepsilon' = \varepsilon\frac{n}{n+(3km-kn)} = \frac{1}{1+(3f(k)-k)}$ if $d^j = f(k)$
$\Rightarrow$ MAXW$\mathbf{P}_k$PACKING is $\mathbf{APX}-\mathbf{h}$ in (planar) bipartite max deg 3
for any $k \geq (k=3)$

3. MIN3-PATHPART. & APX

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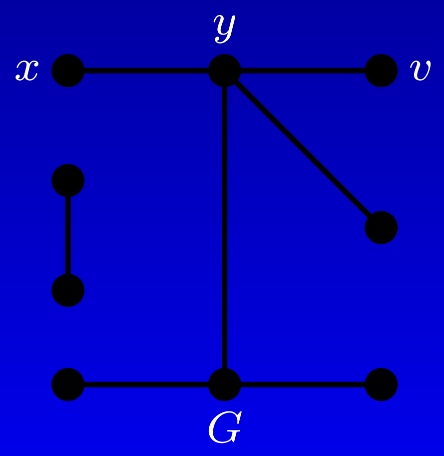

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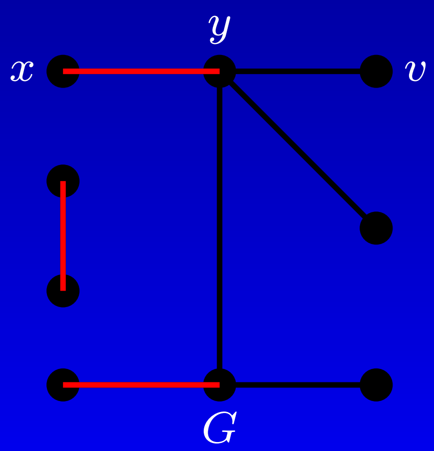

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2. Build the bipartite graph $G_2 = (L, R; E_2)$ defined as:

$$L \equiv M_1,\ R \equiv V \setminus V(M_1)$$

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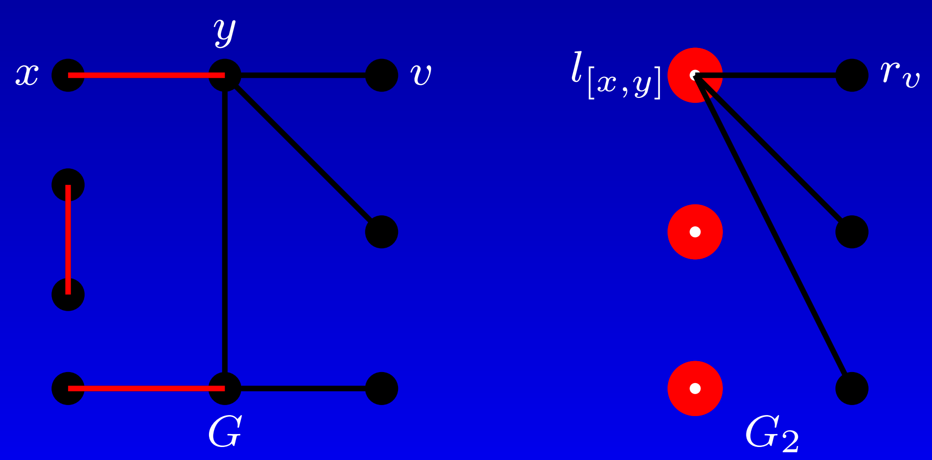

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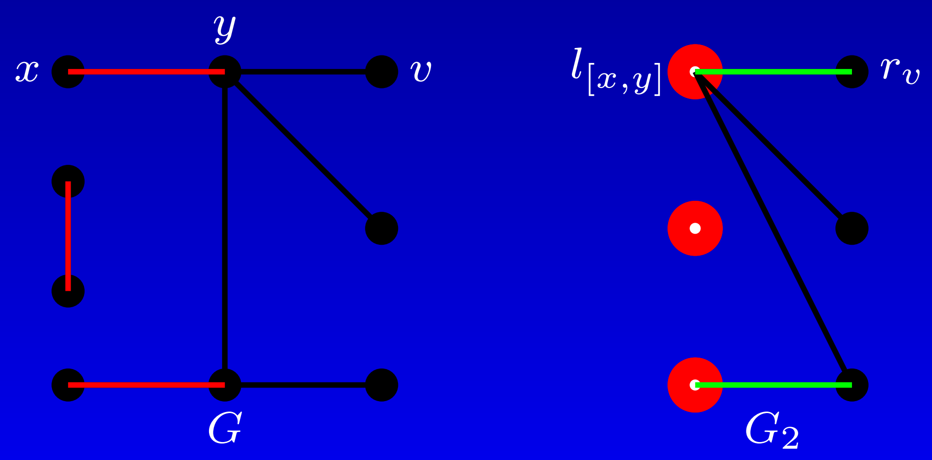

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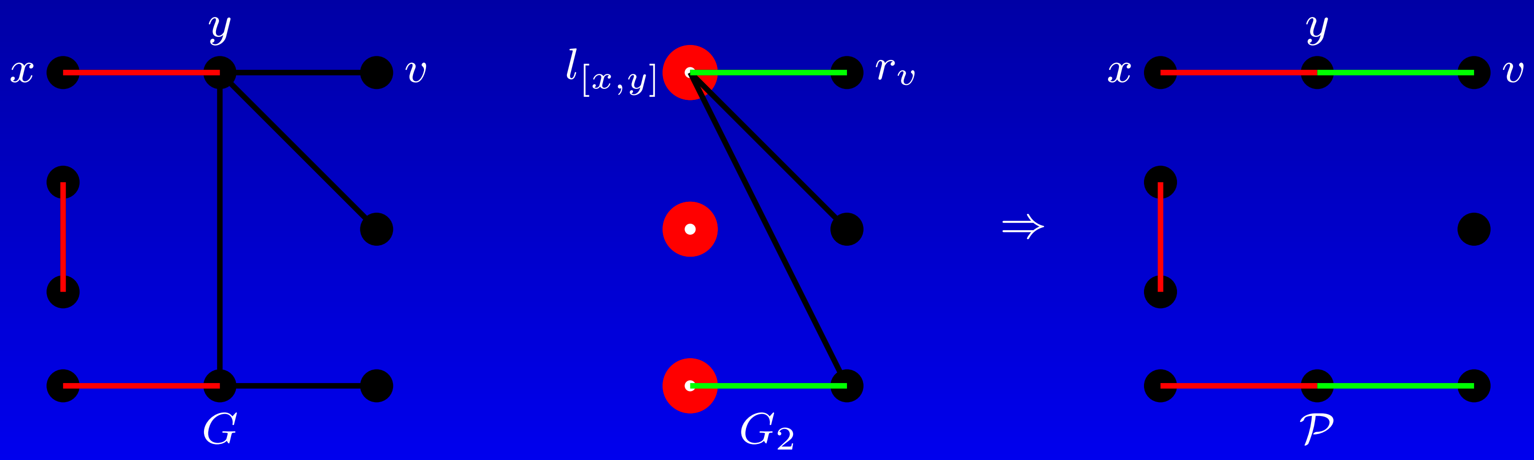
Output $\mathcal{P}$ deduced from M_1, M_2 and $V \setminus V(M_1 \cup M_2)$



3. MIN3-PATHPART. & APX

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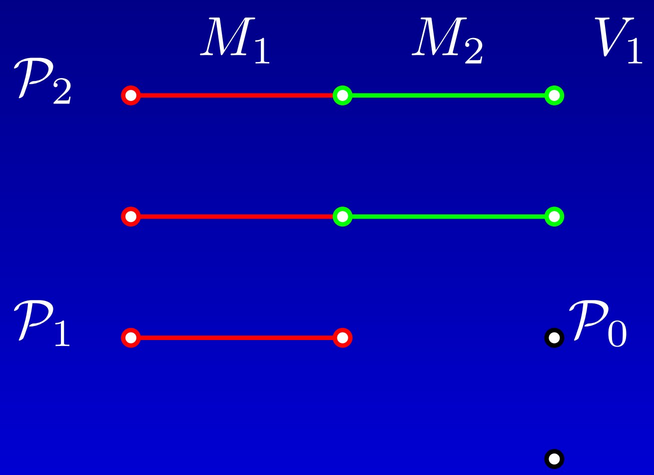
$\mathcal{P} \longrightarrow \{\mathcal{P}_2, \mathcal{P}_1, \mathcal{P}_0\}$ (approximate) $\quad \mathcal{P}^* \longrightarrow \{\mathcal{P}_2^*, \mathcal{P}_1^*, \mathcal{P}_0^*\}$ (optimum)



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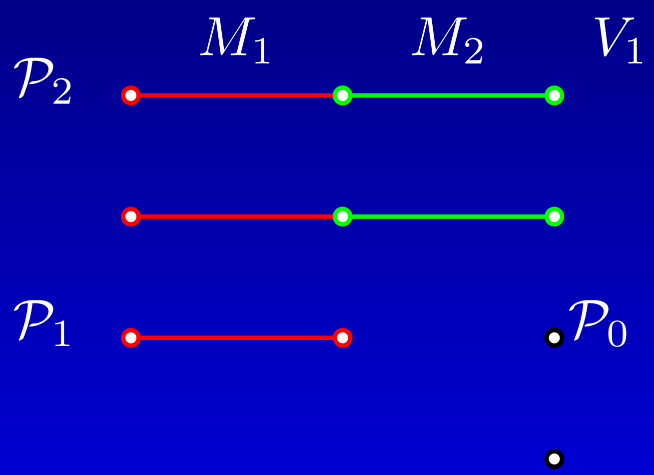

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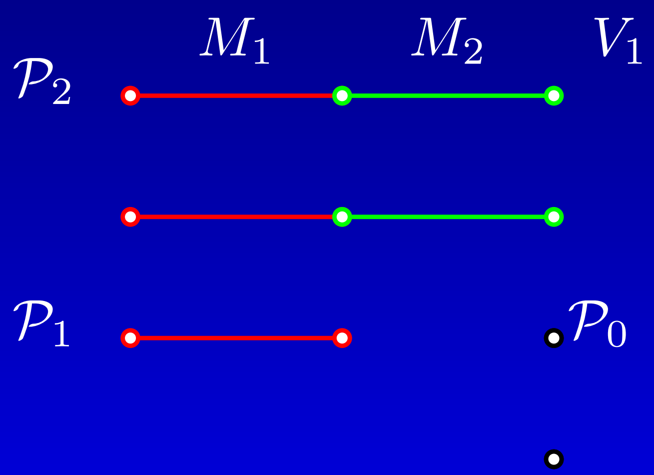

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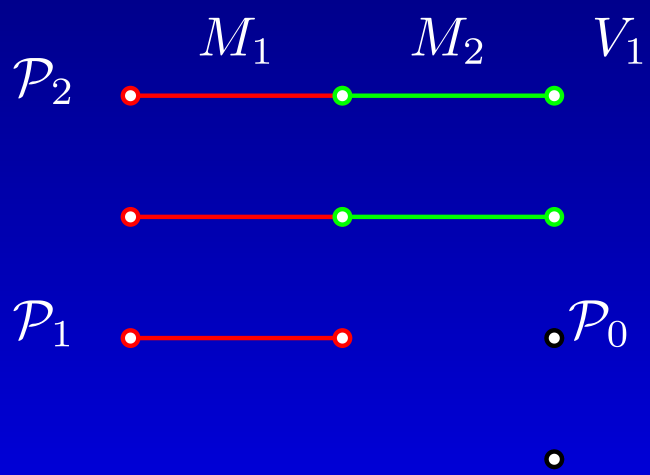

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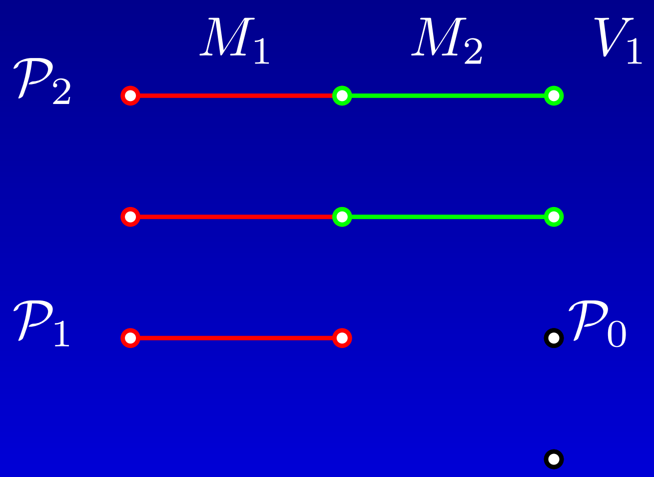

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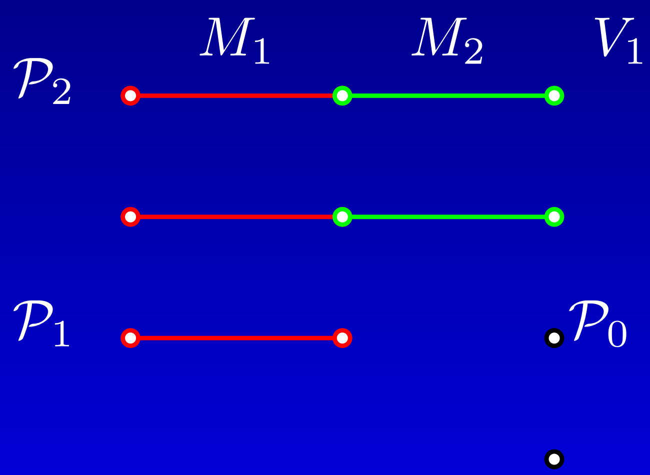
4. $v(\mathcal{P}) = |M_1| + |V_1| - |\mathcal{P}_2|$



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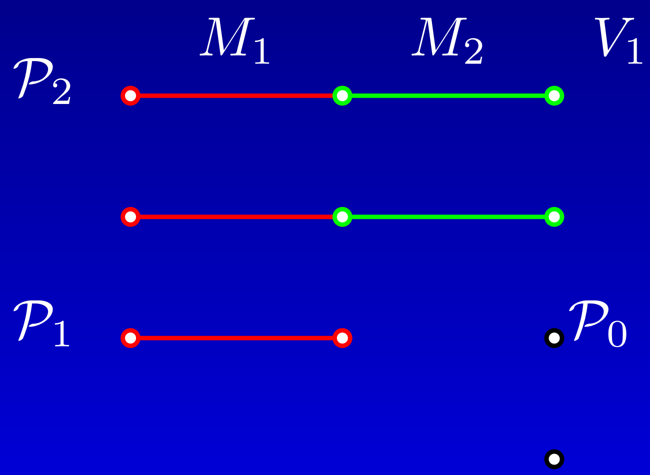
$\mathcal{P} \longrightarrow \{\mathcal{P}_2, \mathcal{P}_1, \mathcal{P}_0\}$ (approximate) $\quad \mathcal{P}^* \longrightarrow \{\mathcal{P}_2^*, \mathcal{P}_1^*, \mathcal{P}_0^*\}$ (optimum)

$V_1 = V \setminus V(M_1) \quad V_2 = V_1 \setminus \mathcal{P}_0^*$

$\Rightarrow$

1. $|V| = 3|\mathcal{P}_2^*| + 2|\mathcal{P}_1^*| + |\mathcal{P}_0^*|$
2. $|V| = 2|M_1| + |V_1|$
3. $v(\mathcal{P}^*) = |\mathcal{P}_2^*| + |\mathcal{P}_1^*| + |\mathcal{P}_0^*|$
4. $v(\mathcal{P}) = |M_1| + |V_1| - |\mathcal{P}_2|$
5. $|V_1| - |V_2| \leq |\mathcal{P}_0^*|$



3. MIN3-PATHPART. & APX

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$3v(\mathcal{P}^*) \geq |V| + |\mathcal{P}_0^*|$ (from (1)+(3))

$$\left.\begin{array}{l} 1.\ |V| = 3|\mathcal{P}_2^*| + 2|\mathcal{P}_1^*| + |\mathcal{P}_0^*| \\ 3.\ v(\mathcal{P}^*) = |\mathcal{P}_2^*| + |\mathcal{P}_1^*| + |\mathcal{P}_0^*| \end{array}\right\} \Rightarrow 3v(\mathcal{P}^*) = |V| + |\mathcal{P}_1^*| + 2|\mathcal{P}_0^*|$$

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$$\left.\begin{array}{l} 2.\ |V| = 2|M_1| + |V_1| \\ 4.\ v(\mathcal{P}) = |M_1| + |V_1| - |\mathcal{P}_2| \\ 5. |V_1| - |V_2| \leq |\mathcal{P}_0^*| \\ 6. |\mathcal{P}_2| \geq 1/2|V_2| \end{array}\right\} \Rightarrow \begin{array}{rcl} 2v(\mathcal{P}) & = & |V| + |V_1| - 2|\mathcal{P}_2| \\ & \leq & |V| + |V_1| - |V_2| \\ & \leq & |V| + |\mathcal{P}_0^*| \end{array}$$

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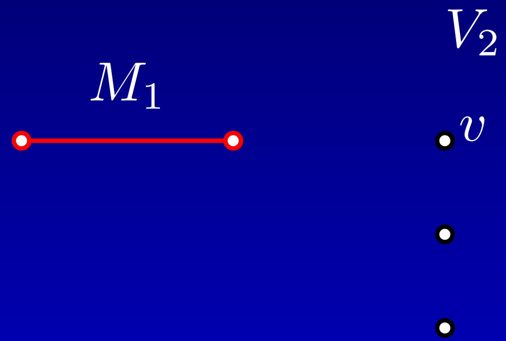

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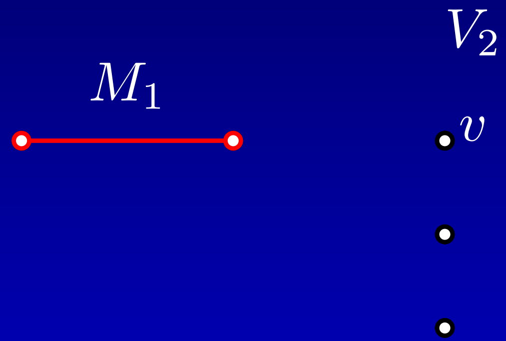

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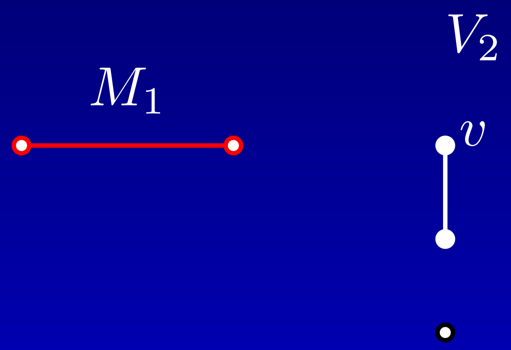

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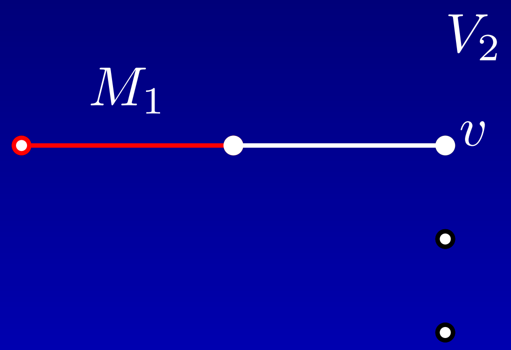

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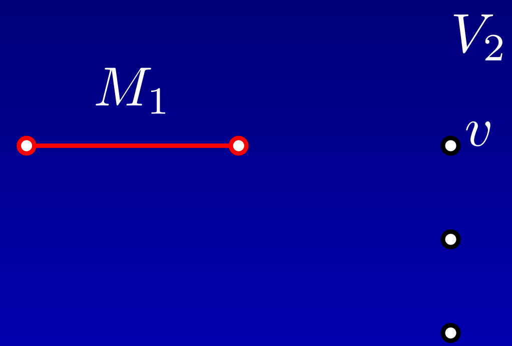

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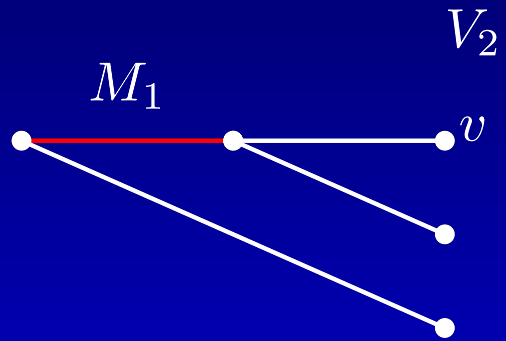

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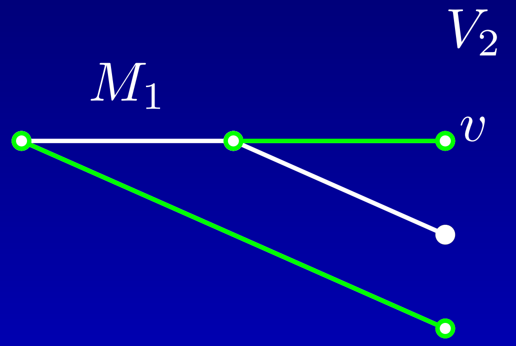

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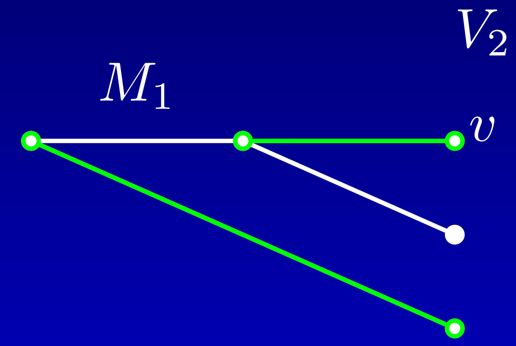

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$\Rightarrow$ E_2 contains a matching of size at least $|V_2|/2$

4. Perspectives

P vs. **NP-c**

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PTAS or not **PTAS**?

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