Increasing Machine Speed in On-line Scheduling of Unit-length Jobs in Slotted Time

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Introduction

We study a problem known as *packet switching, buffer management with bounded delay*:

- Input: non-empty set of jobs with:
    - release time, deadline (integers)
    - weight (also called value)
- Execution of any job takes one unit of time
- Jobs must be executed one at a time
- Goal: to maximize the total weight of executed jobs

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This is the *off-line* version of the problem – the complete input is made available to the algorithm immediately.
In this version the optimal solution can be found easily (polynomial time).

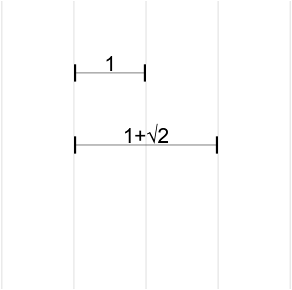
On-line version

More common scenario – there is no information about the future.
In the *on-line* version:

- At each step the algorithm makes a decision which job to execute
- The jobs become "visible" after their respective release times
- Each decision is irrevocable

In the on-line setting the algorithm seems to have a clear disadvantage compared to the off-line setting.

Example



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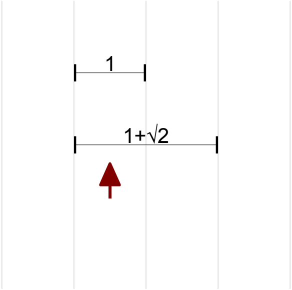

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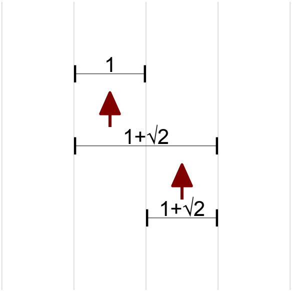

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Competitive ratio

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But there is a better lower bound.

Lower bound

Theorem (Hajek 2001)

Every on-line algorithm has a competitive ratio at least equal to $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

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The proof uses a remarkably simple class of jobs – with lengths at most 2.
Consequently this lower bound holds also for many restricted versions of the problem.

Upper bounds

Progress in recent years:

- 2 (Kesselman et al. 2001, Hajek 2001)
- $\frac{64}{33} \approx 1.939$ (Chrobak et al. 2004)
- 1.852... (Li et al. 2007)
- $2\sqrt{2} - 1 \approx 1.828$ (Englert and Westermann 2007)

Agreeable deadlines

An interesting restriction of the problem: *agreeable deadlines*.

Definition

We say that the jobs forming the set S have agreeable deadlines if and only if

$$\forall i, j \in S : r_i < r_j \Rightarrow d_i \leqslant d_j$$

In other words – the availability interval of one job is not contained in the interior of the availability interval of another job.

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What about the upper bound?

Theorem (Li et al. 2005)

There exists an algorithm having a competitive ratio exactly $\phi \approx 1.618$ in the agreeable deadlines setting.

Resource augmentation

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The modification

The on-line algorithm may now execute more than one job per time slot, given by the parameter k.

The “quick” on-line algorithm is compared to the “slow” off-line algorithm using the competitive ratio.
Our task is to find some lower and upper bounds for this ratio (depending on k).

Upper bound

Natural first choice: greedy algorithm.

Fact

The competitive ratio of the greedy algorithm is equal to $1 + \frac{1}{k}$.

But we can do better than that.

Upper bound

A better algorithm $EG(k)$ is presented below. Let h denote the heaviest available job (note that it may change during the step). In each time slot the algorithm executes:

- The most urgent available job with weight at least $2^{-k}w_h$
- The most urgent available job with weight at least $2^{-k+1}w_h$
- $\dots$
- The most urgent available job with weight at least $2^{-1}w_h$

"Most urgent" means the job whose deadline will be reached next. Ties can be broken in an arbitrary way.

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We define a charging function $c : OPT_1(I) \to \mathbb{Z}$ such that

$$c(j) = min(t_{OPT_1}(j), t_{EG_k}(j))$$

Upper bound

For every time slot t such that $w(c^{-1}(t)) > 0$ we prove that

$$w(c^{-1}(t)) < \left(1 + \frac{1}{2^k - 1}\right) w(t_{EG_k}^{-1}(t))$$

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Thus

$$w(OPT_1(I)) < \left(1 + \frac{1}{2^k - 1}\right) w(EG_k(I))$$

This means that $EG(k)$ is $\left(1 + \frac{1}{2^k-1}\right)$-competitive. It can be shown easily that $EG(k)$ is not competitive for any lower ratio.

Lower bound

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Theorem (J. 2009)

Every k-speed on-line algorithm has a competitive ratio higher than $1 + \varepsilon_k$.

In fact this remains true if we strengthen the algorithm by allowing it to conserve its processing power for the future – we call such an algorithm *cumulative*.

Lower bound – proof outline

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We define a strategy S_k for the adversary. Our goals are:

- Any k-speed cumulative algorithm playing against S_k should execute jobs with total value smaller than the jobs executed by OPT_1 on the same instance
- The length of the game should not exceed l_k
- The total weights of the jobs should not exceed M_k
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Strategy S_1 was defined earlier. Subsequent strategies are generated recursively.

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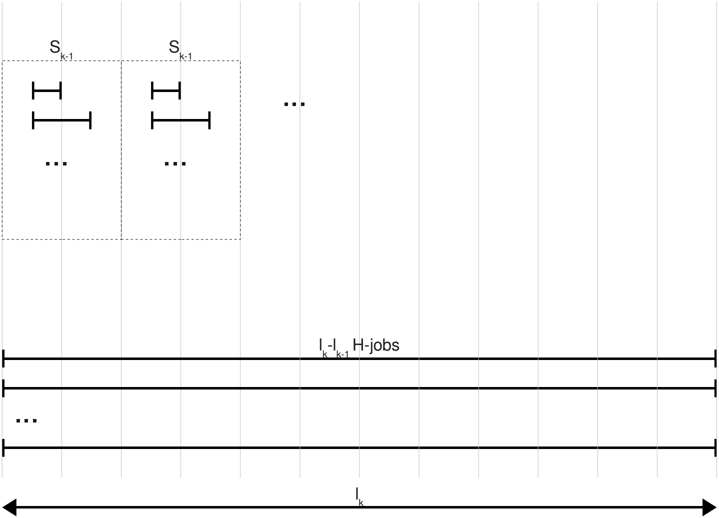

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Otherwise, if the algorithm executes at most $k-1$ L-jobs on average, from the properties of S_{k-1} we know that the algorithm does not perform optimally on the L-jobs from this phase. In this case the adversary ends the game.

Lower bound – proof outline

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$$l_k \leqslant 2^{2^{2k}}$$

$$M_k \leqslant 2^{2^{2^{3(k-1)}}}$$

$$\varepsilon_k \geqslant 1 + \frac{1}{M_k} \geqslant 1 + \left(\frac{1}{2}\right)^{2^{2^{3(k-1)}}}$$

The gap between the lower and upper bound is quite big.

Restriction to agreeable deadlines

Using the normal competitive ratio technique there seems to be not much of a difference between the general case and the restriction to jobs with agreeable deadlines.

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This is different with resource augmentation:

Theorem (J. 2009)

There exists a 2-speed algorithm with competitive ratio 1 in the agreeable deadlines setting.

Open problems

- Find an even broader class of instances where resource augmented on-line algorithms can achieve a comtetitive ratio equal 1
- Reduce the gap between the lower and upper bound in the general k-speed scenario
- Find the best possible competitive ratio for the 1-speed scenario

Thank you

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