

A Problem Kernelization for Graph Packing

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H-Packing

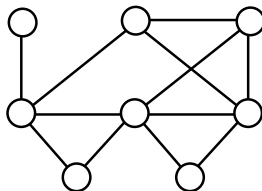
H-Packing Problem

Input: An undirected graph $G = (V, E)$ and a parameter k .

Question: Exist at least k vertex-disjoint copies of H in G ?

Example

$H =$ 




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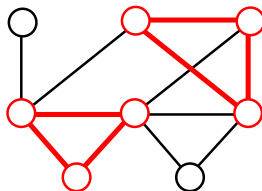
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- ▶ Deterministic algorithm with running time $12.8^{|V(H)|k} \cdot \text{poly}(n)$ [KNEIS, MÖLLE, RICHTER, ROSSMANITH, WG 2006]
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- ▶ H contains three vertices: deterministic algorithm with running time $3.52^{3k} \cdot \text{poly}(n)$ [WANG & FENG, TAMC 2008]
- ▶ H is a path on three vertices: deterministic algorithm with running time $2.48^{3k} \cdot \text{poly}(n)$ [FERNAU & RAIBLE, COCOA 2008]

Known Results (2)

If H is a triangle:

- ▶ Factor- $(3/2 + \epsilon)$ polynomial-time approximation
[HURKENS & SCHRIJVER, SIAM Journal on Discrete Mathematics, 1989]
- ▶ APX-hard in general graphs, factor-1.2 polynomial-time approximation on graphs with maximum degree four
[MANIC & WAKABAYASHI, Discrete Mathematics, 2008]
- ▶ NP-hard to approximate within ratio $139/138$
[CHLEBÍK & CHLEBÍKOVÁ, CIAC 2003]

Fixed-Parameter Tractability

Definition

A problem is *fixed-parameter tractable* with respect to parameter k if it can be solved in $f(k) \cdot \text{poly}(n)$ time.

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Problem Kernel

Parameterized problem L . Instance (I, k) .

$$(I, k) \xrightarrow[\text{poly}(n) \text{ time}]{\text{data reduction rules}} (I', k')$$

- ▶ $(I, k) \in L \leftrightarrow (I', k') \in L$
- ▶ $k' \leq k$
- ▶ $|I'| \leq g(k)$

Kernelization Results

Known Results

- ▶ H is a triangle: $O(k^3)$ -vertex kernel
[FELLOWS, HEGGERNES, ROSAMOND, SLOPER, TELLE, WG 2004]
- ▶ H is a path on three vertices: $7k$ -vertex kernel
[WANG, NING, FENG, CHEN, TAMC 2008]
- ▶ H is a star with d leaves: $((d^3 + 4d^2 + 6d + 4) \cdot k)$ -vertex kernel
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Conjecture: K_t -PACKING might be a problem whose kernel cannot be smaller than $O(k^t)$ vertices.

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New Result

- ▶ $O(k^{|V(H)|-1})$ -vertex kernel for H -PACKING

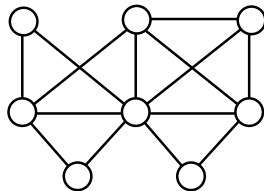
Kernel for Triangle Packing

Data Reduction Rule

Remove all vertices and edges that are not contained in any triangle in G .

Witness Structure \mathcal{T}

A *witness structure* is a maximal set \mathcal{T} of triangles in G that pairwise intersect in at most one vertex.



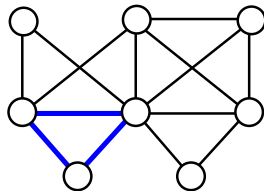
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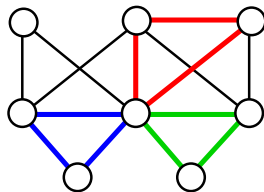
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Bounding the Size of \mathcal{T}

Data Reduction Rule

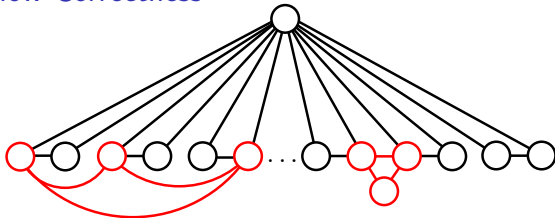
If there is a vertex that is contained in at least $3k - 2$ triangles in \mathcal{T} , then delete it from G and set $k := k - 1$.

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Idea to Show Correctness

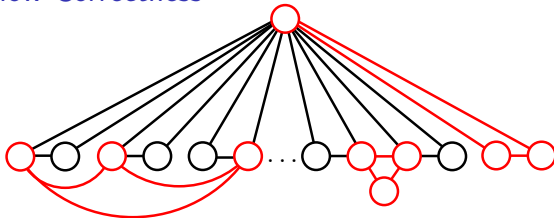


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Bounding the Size of \mathcal{T}

Lemma

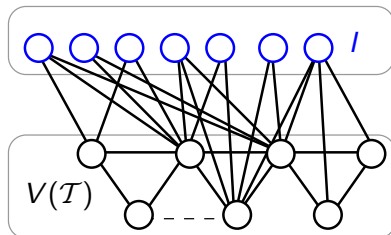
The witness structure \mathcal{T} contains $O(k^2)$ triangles.

Proof

- ▶ Suppose that there is no packing of k triangles and let P be a maximum packing of triangles.
- ▶ P contains at most $k - 1$ triangles.
- ▶ Each vertex of each triangle is contained in at most $3k - 3$ triangles of \mathcal{T} .
- ▶ There are at most $3 \cdot (k - 1) \cdot (3k - 3) = O(k^2)$ triangles in \mathcal{T} .

Bounding the Size of the Remaining Graph

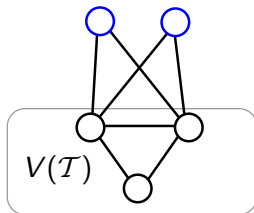
Structure



- ▶ $I := V \setminus V(\mathcal{T})$ forms an independent set.
- ▶ Each triangle that contains a vertex in I shares two vertices with a triangle in \mathcal{T} .

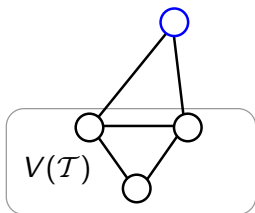
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General Idea



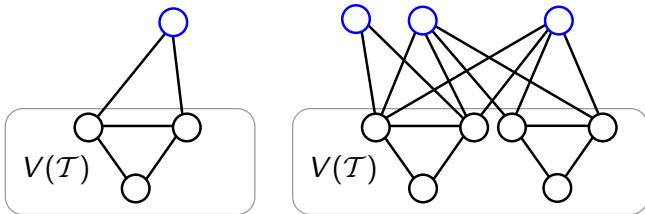
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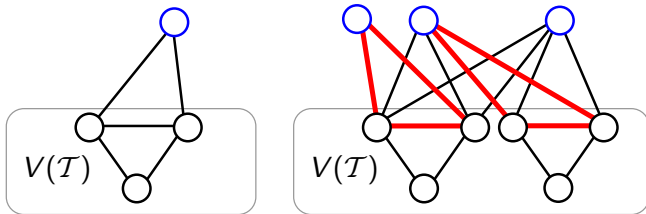
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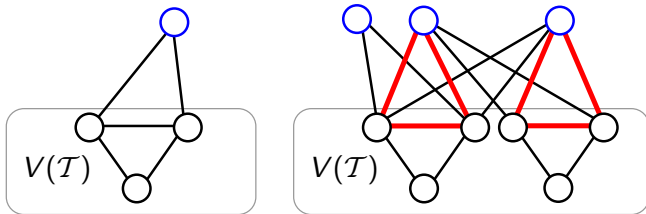
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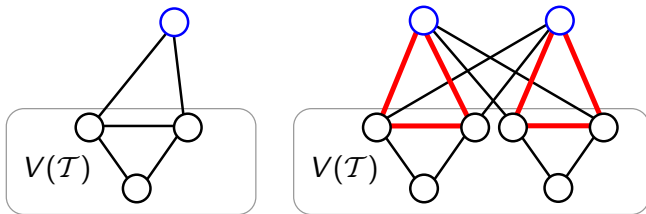
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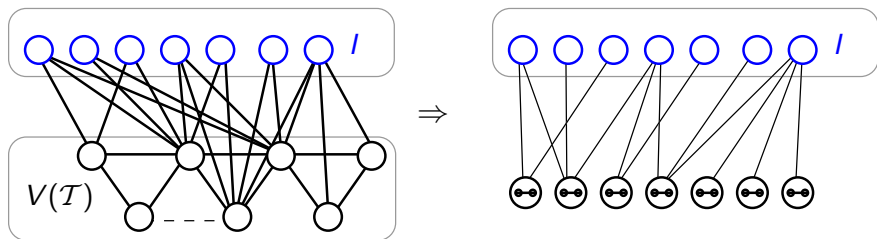
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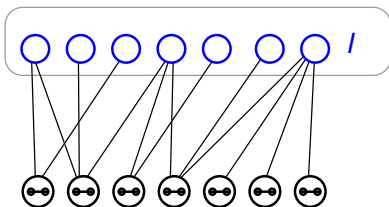
Auxiliary Graph



Bounding the Size of the Remaining Graph

Data Reduction Rule

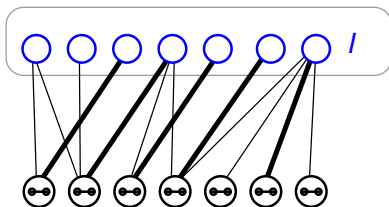
Compute a maximum matching in the auxiliary graph and remove all unmatched vertices in I .



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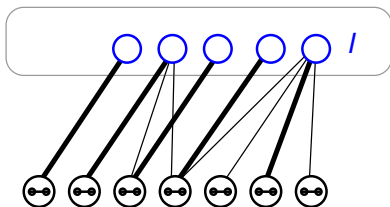
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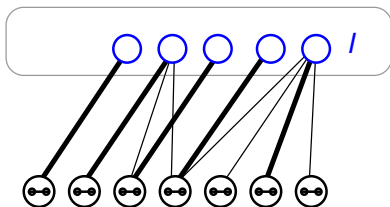
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After removing the unmatched vertices, there are no more vertices in I than edges in triangles in \mathcal{T} .

\Rightarrow There are $O(k^2)$ vertices in I .

Summary

- ▶ The witness structure contains $O(k^2)$ triangles.
- ▶ There are $O(k^2)$ vertices contained in these triangles.
- ▶ The number of remaining vertices is bounded by $O(k^2)$.
- ▶ All the steps can be performed in polynomial time.
- ▶ Thus, we obtain a $O(k^2)$ -vertex kernel.

Further Result and Open Questions

Further Result

- ▶ The approach can be generalized to H -Packing.

Open Questions

- ▶ Does TRIANGLE PACKING admit a $size-O(k^2)$ kernel?
- ▶ “Ultimate Goal”: Does TRIANGLE PACKING admit an $O(k)$ -vertex kernel?

Thank you!