

# Unambiguous Erasing Morphisms in Free Monoids

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# Motivation

- ▶ Systematic research on the (un)ambiguity of morphisms initiated in [FRS06] (Freydenberger, Reidenbach, Schneider, “Unambiguous Morphic Images Of Strings”, *Int. Journal of Foundations of Computer Science* 17, 2006)
- ▶ [FRS06] characterises strings with an unambiguous image of an *nonerasing* morphism.
- ▶ Only few results are known when regarding *all* (possibly erasing) morphisms.

# Basic Definitions and Notations

- ▶ the set of natural numbers  $\mathbb{N} := \{1, 2, \dots\}$ , also used as infinite alphabet
- ▶  $\alpha \in \mathbb{N}^*$ : a string over  $\mathbb{N}$ , called *pattern*
- ▶  $\text{var}(\alpha)$ : symbols in  $\alpha$  (*variables*)
- ▶  $|\alpha|$ : length of  $\alpha$
- ▶  $\varepsilon$ : the empty word

## Example

$\alpha := 2 \cdot 5 \cdot 25 \cdot 25 \cdot 5 \in \mathbb{N}^*$ ,  $\text{var}(\alpha) = \{2, 5, 25\}$ ,  $|\alpha| = 5$ .

## Basic Definitions and Notations (2)

- ▶ a **morphism**: mapping  $h : \mathcal{A}^* \rightarrow \mathcal{B}^*$ ,  $\mathcal{A}, \mathcal{B}$  alphabets, compatible with concatenation, i. e., for all  $v, w \in \mathcal{A}^*$ ,  $h(vw) = h(v)h(w)$ .
- ▶ primarily focus on
  - ▶ morphisms  $\sigma : \mathbb{N}^* \rightarrow \Sigma^*$  with a (possibly finite) target alphabet  $\Sigma$ , e.g.  $\Sigma = \{a, b\}$ ,
  - ▶ (endo)morphisms  $h : \mathbb{N}^* \rightarrow \mathbb{N}^*$ .

# (Un)ambiguous Morphisms

## Definition

$\Sigma$  alphabet,  $\sigma : \mathbb{N}^* \rightarrow \Sigma^*$  morphism,  $\alpha \in \mathbb{N}^+$ ,  $\sigma(\alpha) \neq \varepsilon$ .

We call  $\sigma$  **unambiguous (with respect to  $\alpha$ )** if and only if there is no morphism  $\tau : \mathbb{N}^* \rightarrow \Sigma^*$  satisfying

- ▶  $\tau(\alpha) = \sigma(\alpha)$ ,
- ▶ for some  $x \in \text{var}(\alpha)$ ,  $\tau(x) \neq \sigma(x)$ .

If  $\sigma$  is not unambiguous with respect to  $\alpha$ , it is called **ambiguous (with respect to  $\alpha$ )**.

$\text{AMB}_\Sigma := \{\alpha \in \mathbb{N}^+ \mid \text{there is no unambiguous morphism } \sigma : \mathbb{N}^* \rightarrow \Sigma^* \text{ with respect to } \alpha\}$ .

# The Main Question

Given a pattern  $\alpha \in \mathbb{N}^+$  and a (target) alphabet  $\Sigma$ .

- ▶ Does there exist an unambiguous morphism  $\sigma : \mathbb{N}^* \rightarrow \Sigma^*$  with respect to  $\alpha$ ???

## An Example

Let  $\alpha := 1 \cdot 2 \cdot 2$ .

Let  $\sigma_1, \sigma_2 : \mathbb{N}^* \rightarrow \{a, b\}^*$  be morphisms, given by

$$\sigma_1(1) = a, \sigma_1(2) = b,$$

$$\sigma_2(1) = a, \sigma_2(2) = \varepsilon.$$

- ▶  $\sigma_1$  is **ambiguous** with respect to  $\alpha$  since  $\tau : \mathbb{N}^* \rightarrow \{a, b\}^*$ , given by
 
$$\tau(1) = a b b, \tau(2) = \varepsilon$$
 satisfies
 
$$\sigma(\alpha) = a b b = \tau(\alpha)$$
 and
 
$$\sigma(1) \neq \tau(1).$$
- ▶  $\sigma_2$  is **unambiguous** with respect to  $\alpha$ .

## Nonerasing Morphisms

In [FRS06] primal focus on **nonerasing** morphisms, i.e. morphisms  $\sigma : \mathbb{N}^* \rightarrow \Sigma^*$  with  $|\sigma(x)| \geq 1$  for all  $x \in \mathbb{N}$ .

Important partition of  $\mathbb{N}^*$  into **succinct** and non-succinct = **prolix** patterns.

### Theorem ([FRS06,RS07])

Let  $\alpha \in \mathbb{N}^*$ .  $\alpha$  succinct  $\iff \alpha$  is the shortest generator of its respective pattern language  $\iff \alpha$  is not a fixed point of a nontrivial morphism  $h : \mathbb{N}^* \rightarrow \mathbb{N}^* \iff \alpha$  is morphically primitive.

Previous example pattern  $\alpha = 1 \cdot 2 \cdot 2$  is *prolix* since there exist a nontrivial morphisms  $h : \mathbb{N}^* \rightarrow \mathbb{N}^*$ ,  $h(1) = 1 \cdot 2 \cdot 2$ ,  $h(2) = \varepsilon$ , which satisfies  $h(\alpha) = \alpha$ .

# Characterisation of Unambiguous Nonerasing Morphisms

## Theorem ([FRS06])

*Let  $\alpha \in \mathbb{N}^*$ , let  $\Sigma$  be an alphabet,  $|\Sigma| \geq 2$ . There exists an unambiguous nonerasing morphism  $\sigma : \mathbb{N}^* \rightarrow \Sigma^*$  with respect to  $\alpha$  if and only if  $\alpha$  is succinct.*

Consequence: If  $\alpha$  not succinct, thus, prolix, and  $\sigma$  unambiguous with respect to  $\alpha$ , then  $\sigma$  is **erasing**, i.e.  $\sigma(x) = \varepsilon$  for an  $x \in \text{var}(\alpha)$ .

$\Rightarrow$  see example pattern...

## Another Example

$$\alpha = 1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 3$$

$\alpha$  *prolix* since there exist a nontrivial morphisms  $h : \mathbb{N}^* \rightarrow \mathbb{N}^*$ ,  
 $h(1) = \varepsilon$ ,  $h(2) = 1 \cdot 2$ ,  $h(3) = 3$ , which satisfies  $h(\alpha) = \alpha$ .

## Another Example

$E = \emptyset$  ← variables to be erased by an unambiguous morphism

$N = \{1, 2, 3\}$  ←  $\text{var}(\alpha) \setminus E$

$$\alpha = 1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 3$$

## Another Example

$$E = \emptyset$$

$$N = \text{var}(\alpha) = \{1, 2, 3\}$$

$$\alpha = 1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 3$$

Assume  $\sigma$  is an unambiguous morphism with respect to  $\alpha$  and  $\sigma(1) \neq \varepsilon$ .

Then  $\tau$  with  $\tau(1) = \varepsilon$ ,  $\tau(2) = \sigma(1 \cdot 2)$ ,  $\tau(3) = \sigma(3)$  satisfies

$$\tau(\alpha) = \sigma(\alpha) \quad \text{and} \quad \tau(1) \neq \sigma(1).$$

$\Rightarrow \sigma$  is ambiguous, contradiction. Thus, 1 must be erased by an unambiguous morphism.

## Another Example

$$E = \{1\}$$

$$N = \{2, 3\}$$

$$\alpha = 1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 3$$

Assume  $\sigma$  is an unambiguous morphism with respect to  $\alpha$  and  $\sigma(1) = \varepsilon$ ,  $\sigma(2) \neq \varepsilon$ .

Then  $\tau$  with  $\tau(1) = \sigma(2)$ ,  $\tau(2) = \varepsilon$ ,  $\tau(3) = \sigma(3)$  satisfies

$$\tau(\alpha) = \sigma(\alpha) \quad \text{and} \quad \tau(1) \neq \sigma(1).$$

$\Rightarrow \sigma$  is ambiguous, contradiction. Thus, 2 must be erased by an unambiguous morphism, too.

## Another Example

$$E = \{1, 2\}$$

$$N = \{3\}$$

$$\alpha = 1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 3$$

Assume  $\sigma$  is an unambiguous morphism with respect to  $\alpha$  and  $\sigma(1, 2) = \varepsilon$ ,  $\sigma(3) \neq \varepsilon$ .

Then  $\tau$  with  $\tau(1) = \varepsilon$ ,  $\tau(2) = \sigma(3)$ ,  $\tau(3) = \varepsilon$  satisfies

$$\tau(\alpha) = \sigma(\alpha) \quad \text{and} \quad \tau(2) \neq \sigma(2).$$

$\Rightarrow \sigma$  is ambiguous, contradiction. Thus, 3 must be erased by an unambiguous morphism, too.

## Another Example

$$E = \text{var}(\alpha) = \{1, 2, 3\}$$

$$N = \emptyset$$

$$\alpha = 1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 3$$

⇒ No morphism is unambiguous with respect to  $\alpha$ .

# Ambiguity Partitions

## Definition

Let  $\alpha \in \mathbb{N}^+$ . We inductively define an **ambiguity partition (with respect to  $\alpha$ )**:

- (i)  $(\emptyset, \text{var}(\alpha))$  is an ambiguity partition with respect to  $\alpha$ .
- (ii) If  $(E, N)$  is an ambiguity partition with respect to  $\alpha$  and there exists a morphism  $h : \mathbb{N}^* \rightarrow \mathbb{N}^*$  which satisfies  $h(i) \neq i$  for an  $i \in N$  and  $h(\alpha) = \delta_N(\alpha)$  then  $(E', N')$  is an ambiguity partition with

$$E' := E \cup \{x \in N \mid h(x) = \varepsilon\},$$

$$N' := \{x \in N \mid h(x) \neq \varepsilon\}.$$

Note that  $E' \supset E$  and  $N' \subset N$ .

## Ambiguity Partitions (2)

Necessary condition for unambiguous morphisms:

### Theorem

*Let  $\Sigma$  be an alphabet. Let  $\alpha \in \mathbb{N}^+$  and let  $(E, N)$  be an ambiguity partition with respect to  $\alpha$ . Then every morphism  $\sigma : \mathbb{N}^* \rightarrow \Sigma^*$  satisfying  $\sigma(x) \neq \varepsilon$  for an  $x \in E$  is ambiguous with respect to  $\alpha$ .*

Sufficient criterion (no unambiguous morphisms):

### Corollary

*Let  $\Sigma$  be an alphabet. Let  $\alpha \in \mathbb{N}^+$ . If  $(\text{var}(\alpha), \emptyset)$  is an ambiguity partition with respect to  $\alpha$ , no morphism  $\sigma : \mathbb{N}^* \rightarrow \Sigma^*$  is unambiguous with respect to  $\alpha$ .*

Results are independent on the size of  $\Sigma$ .



## Morphisms With an Infinite Target Alphabet

For infinite target alphabets, the criterion from the previous corollary is characterising:

### Theorem

Let  $\Sigma_\infty$  be an infinite alphabet and let  $\alpha \in \mathbb{N}^+$ . There is no unambiguous morphism  $\sigma : \mathbb{N}^* \rightarrow \Sigma_\infty^*$  with respect to  $\alpha$  if and only if  $(\text{var}(\alpha), \emptyset)$  is an ambiguity partition with respect to  $\alpha$ .

$$\begin{array}{cc} =E & =N \end{array}$$

And we receive a new NP-complete decision problem:

### Theorem

Let  $\Sigma_\infty$  be an infinite alphabet. The problem of deciding  $AMB_{\Sigma_\infty}$  is NP-complete.

## Morphisms With a Finite Target Alphabets

### Theorem

Let  $k \in \mathbb{N}$  and  $\Sigma_k, \Sigma_{k+1}$  be finite alphabets with  $k$  and  $k + 1$  letters, respectively. There exists a pattern  $\alpha \in \mathbb{N}^+$  such that

- (i)  $(\text{var}(\alpha), \emptyset)$  is not an ambiguity partition with respect to  $\alpha$ ,
- (ii) no morphism  $\sigma : \mathbb{N}^* \rightarrow \Sigma_k^*$  is unambiguous with respect to  $\alpha$ ,
- (iii) there exists an unambiguous morphism  $\sigma' : \mathbb{N}^* \rightarrow \Sigma_{k+1}^*$  with respect to  $\alpha$ .

### Corollary

Let  $\Sigma, \Sigma'$  be finite alphabets,  $|\Sigma| < |\Sigma'|$ . Then  $AMB_{\Sigma} \supset AMB_{\Sigma'}$ .

## A New Point of View

Given a pattern  $\alpha \in \mathbb{N}^+$ .

- ▶ What is the minimum size of  $\Sigma$  such that there exists an unambiguous morphism  $\sigma : \mathbb{N}^* \rightarrow \Sigma^*$  with respect to  $\alpha$  if such a morphisms exists at all?

# Outlook

- ▶ Open problem: Is  $AMB_{\Sigma}$  decidable for finite alphabets  $\Sigma$ ?
- ▶ Work in progress: Development of a better understanding of target-alphabet-specific ambiguity phenomena.

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Thank you for your attention!