Improved algorithms for the 2-vertex-disjoint paths problem

Torsten Tholey

Universität Augsburg

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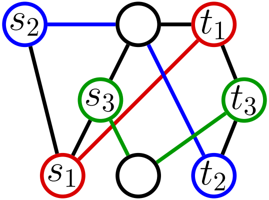
Definitions

k-disjoint paths problem (k-DPP)

Given a graph G and vertices $s_1, \ldots, s_k, t_1 \ldots, t_k$, find k disjoint paths $p_1 : s_1 \to t_1, \ldots, p_k : s_k \to t_k$ if such paths exist.



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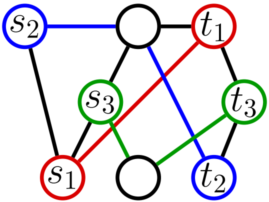
sources and targets

Sources: $s_1, \ldots, s_k$

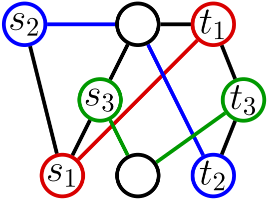
Targets: $t_1, \ldots, t_k$



Applications

Applications of k-disjoint paths problem

- Network reliability,
- VLSI-Design,
- Routing problems.



Previous Results for the k-DPP

Fortune, Hopcroft and Wyllie (1980)

k-DPP is $\mathcal{NP}$-hard for directed graphs.

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Bad News

The algorithms for the k-DPP are not practical.

Results for the 2-DPP

Previous Results

$\mathcal{P}$	Ohtsuki (1980), Seymour (1980), Shiloach (1980), Thomassen (1980).
$O(mn)$	Ohtsuki (1980), Shiloach (1980).
$O(n^2)$	Khuller, Mitchell, Vazirani (1992).
$O(m\alpha(m,n)+n)$	Tholey (2004).

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New Result

$O(m + n\alpha(n,n))$

Results for the 2-DPP on planar graphs

Result of Itai

The problem can be reduced in linear time to triconnected planar graphs.

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$O(m)$	Perl, Shiloach (1978),
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New Result

$O(m)$ simple algorithm for planar graphs without planar embeddings and Itai's reduction.

Hagerup's algorithm on planar graphs

(1) **For** $i := 1$ **to** 2
(2) Construct three disjoint paths $p_1, p_2, p_3 : s_i \rightarrow t_i$.
(3) Let $j \in \{1, 2\}$ such that $i \neq j$.
(4) **For** $k := 1$ **to** 3
(5) **If** there is a path $q : s_j \rightarrow t_j$ in $G - p_k$.
(6) **Return** p_k and q.
(7) **Return** "No paths found".

Generalizing Hagerup's algorithm

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We only need to guarantee the existence of three disjoint paths between s_1 and t_1 as well as between s_2 and t_2.

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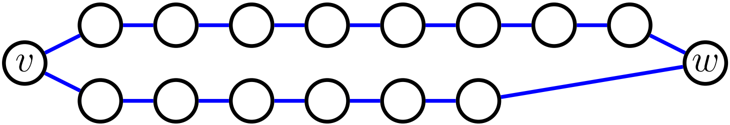
Solution

We split the original instance into smaller instances.

Lemma

Given k disjoint paths $v \to w$ one can find

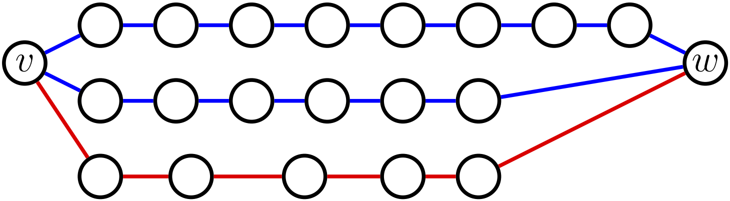


Finding k-separators

Lemma

Given k disjoint paths $v \to w$ one can find

- either a $(k+1)$-th path in $O(m+n)$ time or



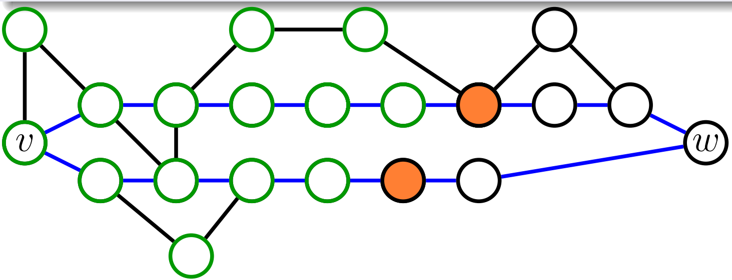
Finding k-separators

Lemma

Given k disjoint paths $v \to w$ one can find

- either a $(k+1)$-th path in $O(m+n)$ time or
- a k-separator separating v and w

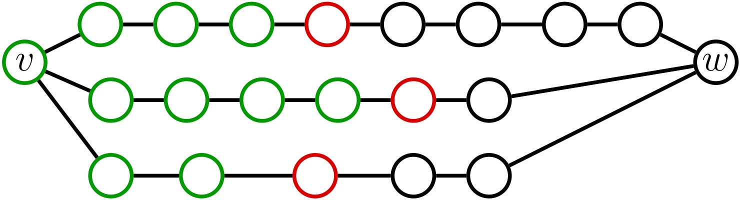
  in time linear in the number of vertices of the connected compontent containing v.



Finding k-separators

not part of a 3-separator

first vertices possibly part of a 3-separator



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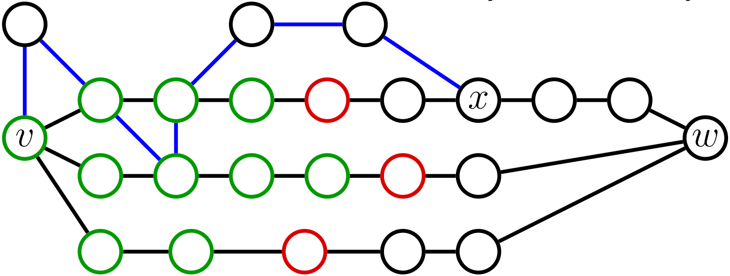

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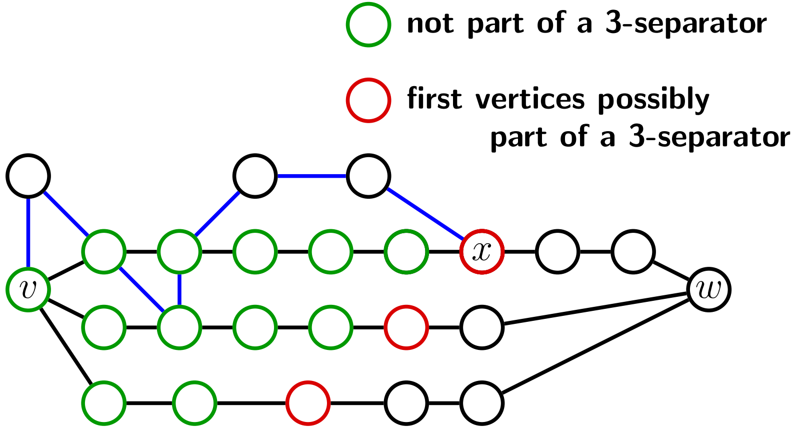
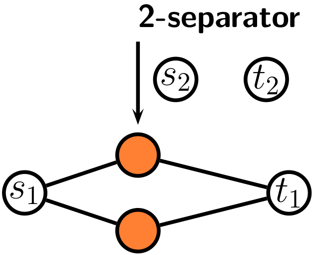
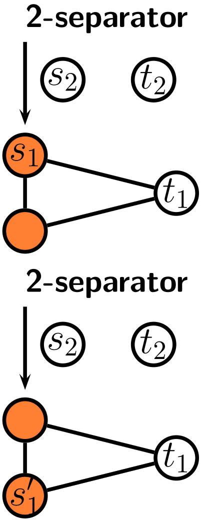


Splitting the Instances





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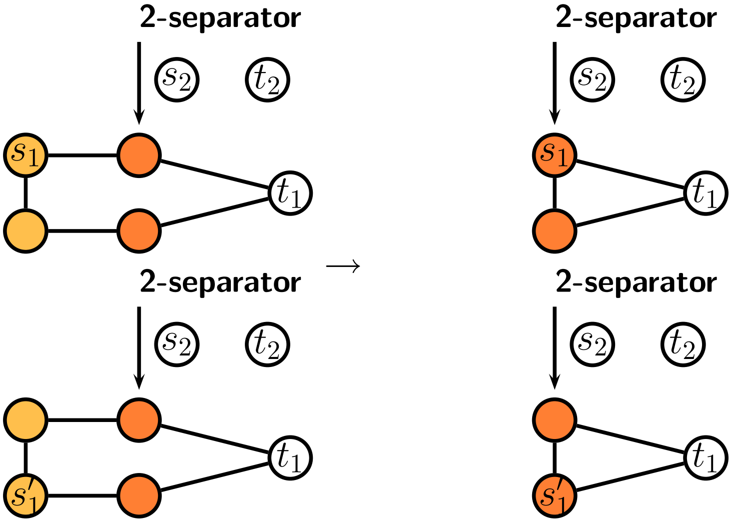

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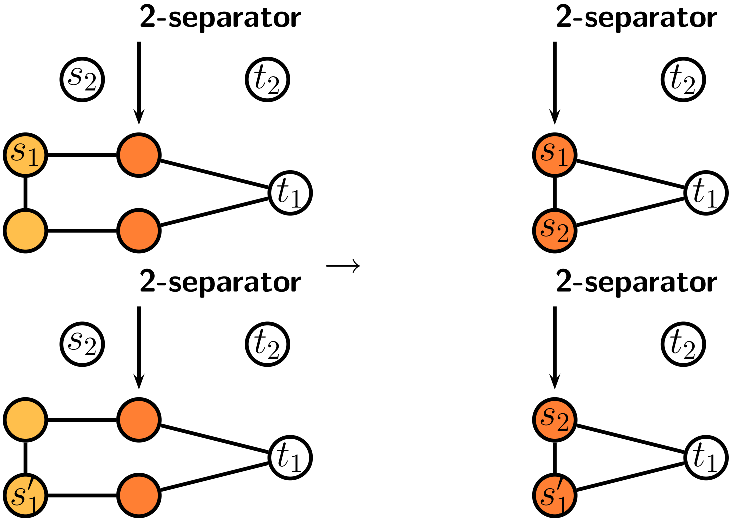

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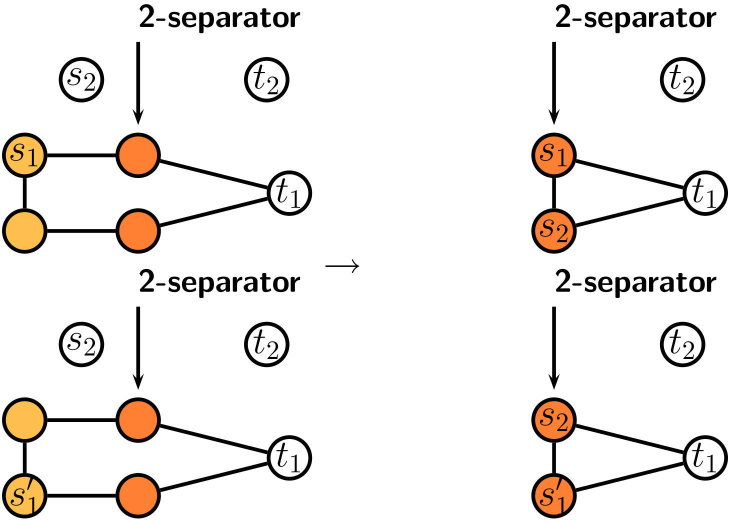
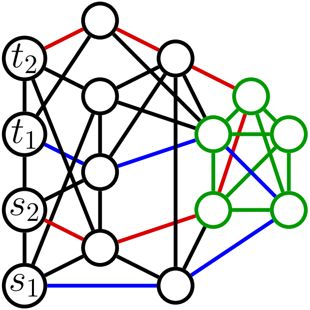


Problem: The instances to solve on the right side depend on a solution of the 2-VDPP for the left part of the left side.

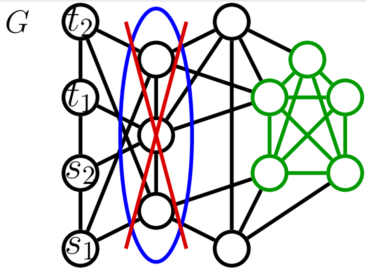
Solution for non-planar graphs



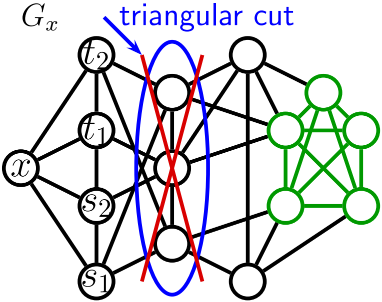
Problem on 3-connected graphs

(P): There are 4 internally disjoint paths from s_1, s_2, t_1, and t_2 to every subset $S \subseteq V$ with $|S| \leq 4$.

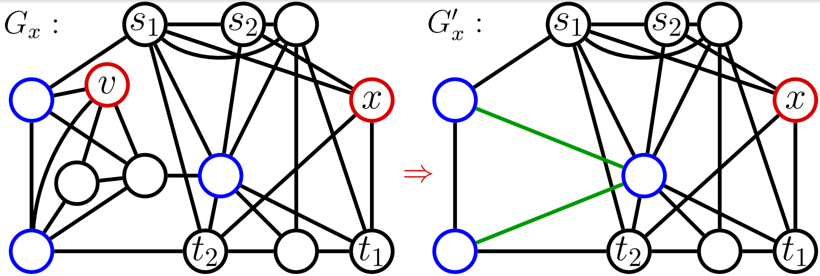
$\Leftrightarrow$



(P*): There is no vertex $v \in G_x$ that is separated from x by a triangular cut.



Solution: Δ-Replacements



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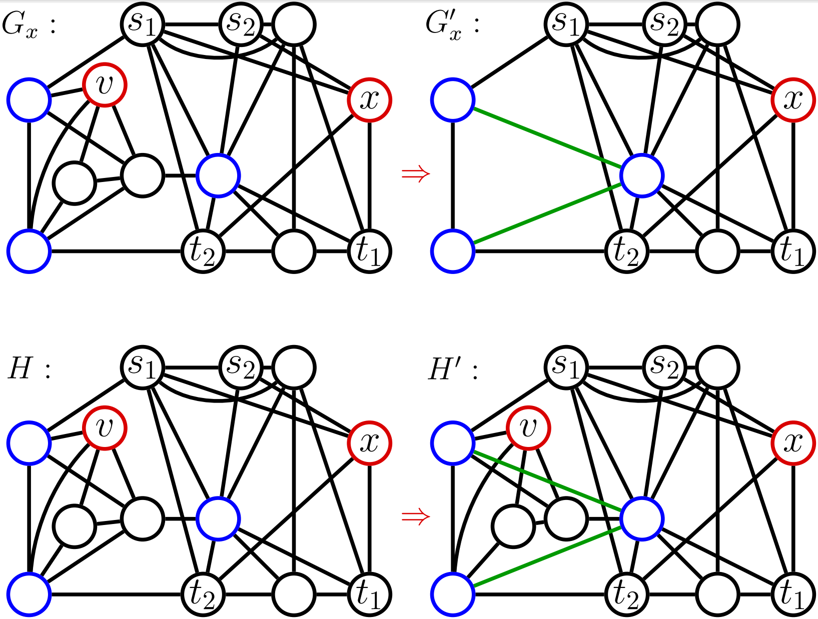

New algorithm

New Idea

Replace H by a sparse certificate for 4-connectivity.

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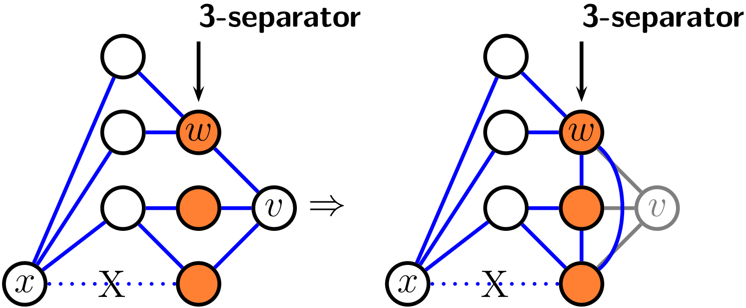
Replace H by a sparse certificate for 4-connectivity.

Sparse certificate for 4-connectivity

K is called a sparse certificate of G if

- $K \subseteq G$.
- Two vertices v and w are 4-connected in K iff the same is true for G.
- $V(K) = V(G)$, $|E(K)| = O(|V(G)|)$.

Connectivity between vertices



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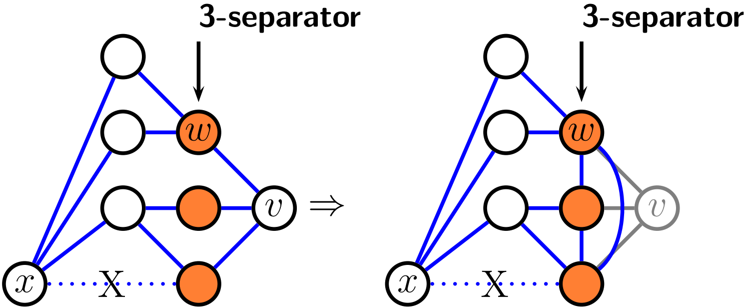
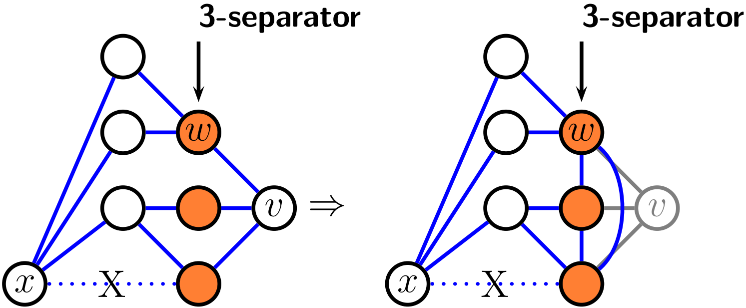


Connectivity between x and w with dotted edges:

3-connected **4-connected**

Connectivity between vertices



Connectivity between x and w with dotted edges:

3-connected 4-connected

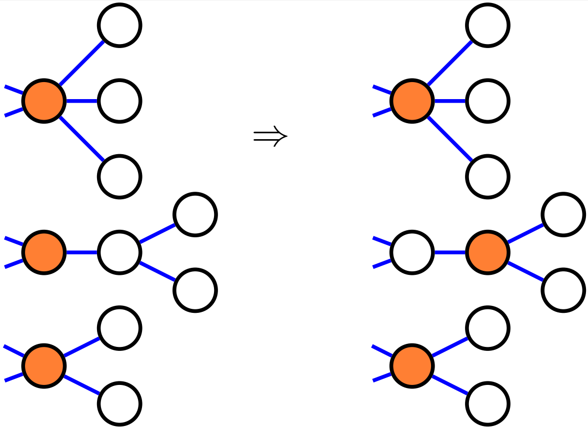
Connectivity between v and w without dotted edges:

3-connected 3-connected

Phase I

Solution

We divide the algorithm in two phases, where the first phase only deletes vertices of degree ≥ 4.



Phase I

Conclusion

At the end of phase I we have $\deg(v) = 3$ for all vertices that are not 4-connected to x.

Open Questions

Most important questions

- Can the 2-DPP be solved in linear time.
- Can edge-disjoint paths on planar graphs also be found in linear time.