# A fast approximation scheme for the multiple knapsack problem

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# Multiple knapsack problem (MKP)

**Given:**

- a set $\mathcal{A}$ of $n$ items with $size(a_j), profit(a_j) \in \mathbb{Z}^+$,
- a set $\mathcal{B}$ of $m$ bins with capacities $c(b) \in \mathbb{Z}^+$.

**Problem:** find a subset $S \subset \mathcal{A}$ of maximum total profit $\sum_{a \in S} profit(a)$ such that $S$ can be packed into $\mathcal{B}$ without exceeding the capacities.

# Results

## Known Results:

- MKP is strongly NP-hard (contains bin packing as special case)
- there is no FPTAS even for two bins (unless $P = NP$) **(Chekuri, Khanna), (Caprara, Kellerer, Pferschy)**
- there is a PTAS for MKP **(Chekuri, Khanna)** with running time

$$n^{O(1/\epsilon^8 \log(1/\epsilon))}.$$

# Table by Downey and Fellows

Tabelle 1: **Running time** of polynomial approximation schemes.

| problem | authors | running time for $\epsilon = 0.2$ |
|---|---|---|
| Euclidean TSP | Arora | $O(\|I\|^{15.000})$ |
| Multiple knapsack | Chekuri and Khanna | $O(\|I\|^{9.275.000})$ |
| Maximum subforest | Shamir and Tsur | $O(\|I\|^{958.267.391})$ |
| General multi. scheduling | Chen and Miranda | $O(\|I\|^{10^{60}})$ |
| Maximum independent set for disk graphs | Erlebach, Jansen and Seidel | $O(\|I\|^{523.804})$ |

# Open Questions for MKP

**(1)** Is there an EPTAS for MKP with an improved running time $f(1/\epsilon)poly(n)$ **(Chekuri, Khanna 2000)**?

**(2)** Is the standard parametrization of MKP W[1]-hard **(Fellows 2003)**?

**Notice:** If the standard parametrization of an optimization problem is W[1]-hard, then the optimization problem does not have an EPTAS (unless FPT=W[1]) **(Bazgan 1995, Cesati and Trevisan 1997)**.

# New Result I

**Theorem (Jansen 2009)**

There is an EPTAS for MKP with running time

$$2^{O(1/\epsilon^5 \log(1/\epsilon))} poly(n).$$

# New Result II

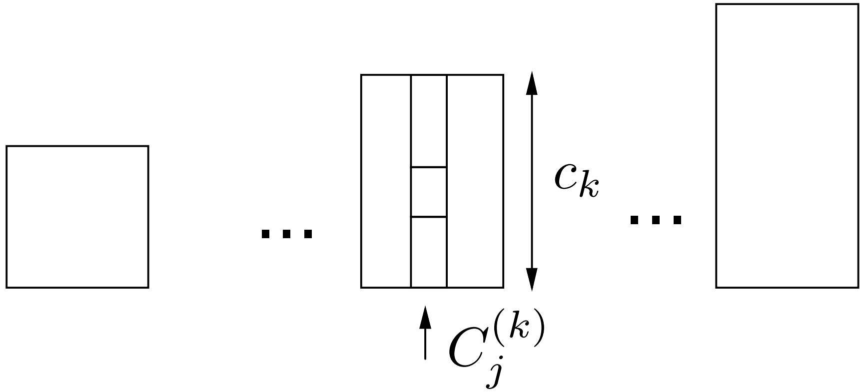
**Theorem (Jansen 2010)**

There is an EPTAS for MKP with running time

$$2^{O(1/\epsilon \log^4(1/\epsilon))} + poly(n).$$

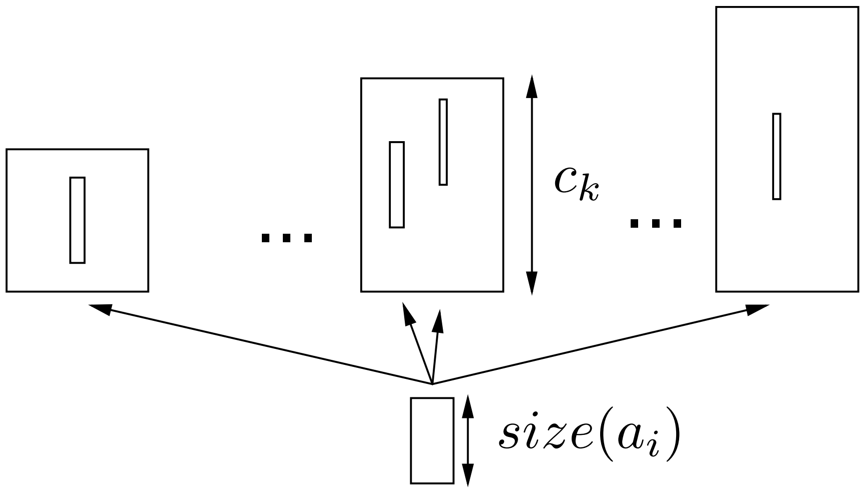
**Note:** Running time can be improved to $2^{O(1/\epsilon \log^2(1/\epsilon))} + poly(n)$ if a **conjecture about the integrality gap for bin packing** is true.

# LP-Relaxation




- a configuration $C_j^{(k)}$ is a subset $S \subset \mathcal{A}$ of items with $\sum_{a \in S} size(a) \leq c_k$.
- use a fractional variable $y_j^{(k)} \in [0,1]$ to denote the length of configuration $C_j^{(k)}$ in the solution.

# LP-Relaxation




- use a variable $x_i \in [0,1]$ to indicate a fractional piece of item $a_i$ and allow this piece to be distributed among the $m$ bins.

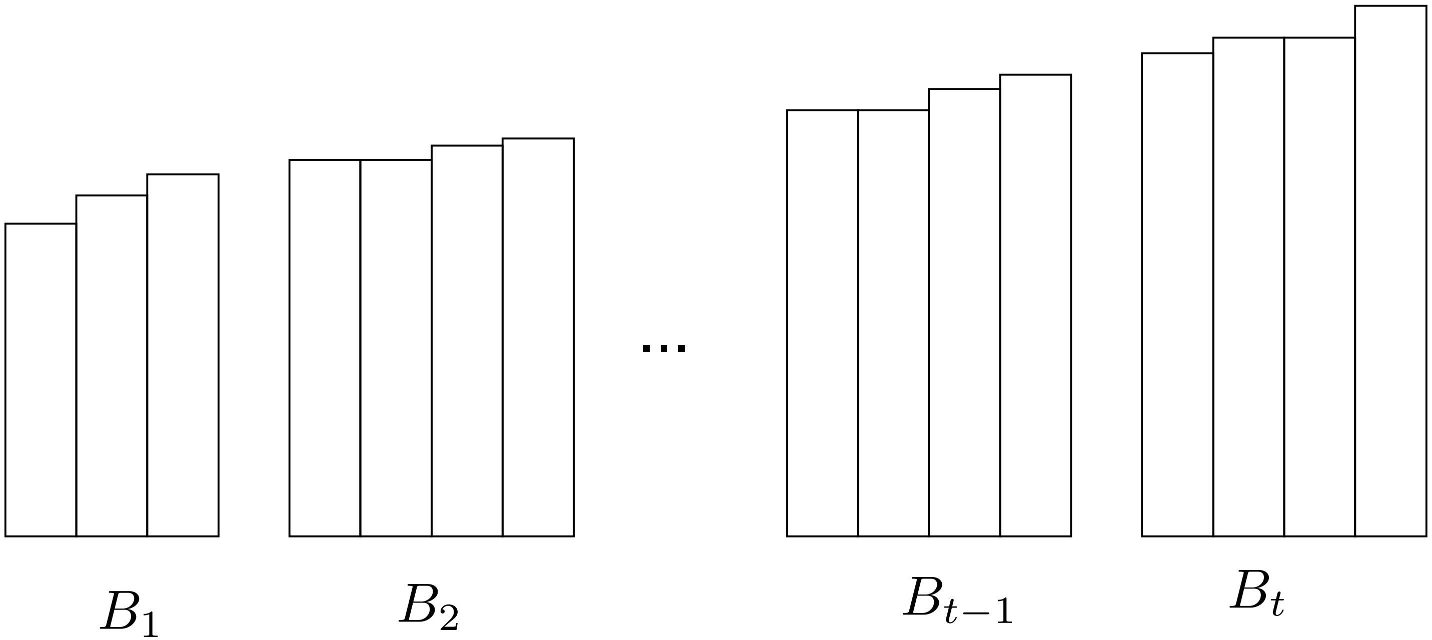
# LP-Relaxation $LP(A, B)$

$$\max \sum_{i=1}^{n} profit(a_i) x_i$$

$$\begin{aligned}
\sum_{k=1}^{m} \sum_{j:a_i \in C_j^{(k)}} y_j^{(k)} &= x_i && \text{for } i = 1, \ldots, n, \\
\sum_{j=1}^{H_k} y_j^{(k)} &\leq 1 && \text{for } k = 1, \ldots, m, \\
y_j^{(k)} &\geq 0 && \text{for } j = 1, \ldots, H_k \\
& && \text{and } k = 1, \ldots, m, \\
x_i &\in [0, 1] && \text{for } i = 1, \ldots, n.
\end{aligned}$$

# Rounding of LP solution I

- Suppose that $m \geq \lceil 1/\delta \log^2(1/\delta) \rceil$ and $\delta = \Theta(\epsilon)$.
- Build blocks with $M = \lceil 1/\delta \log^2(1/\delta) \rceil$ bins with maybe the exception of block $B_1$.



# Rounding the LP-solution II

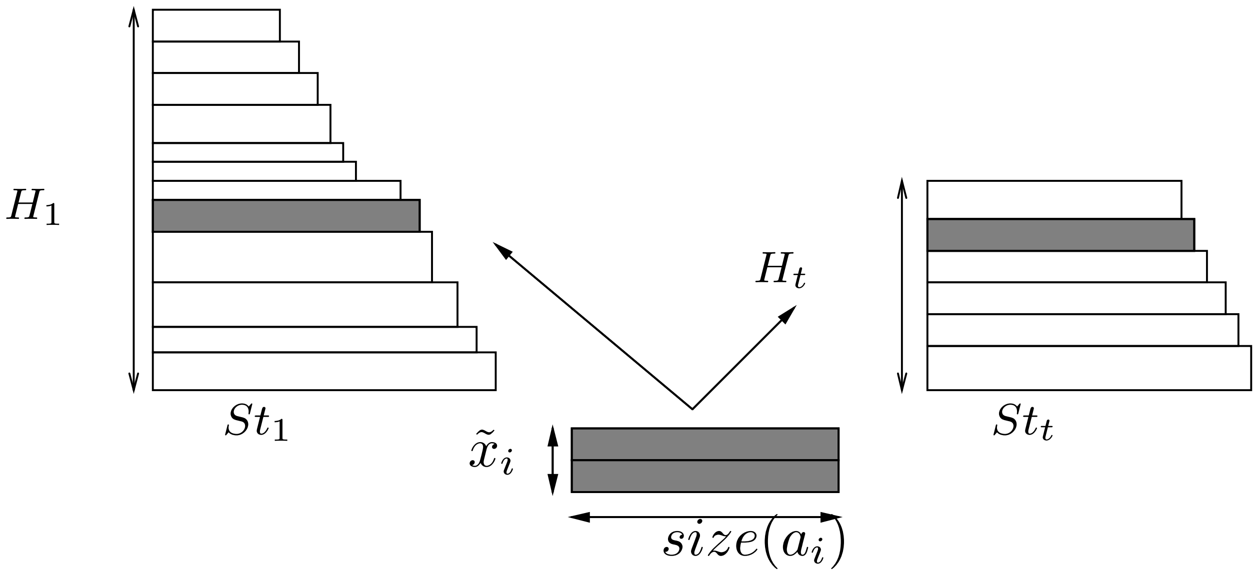
For each item $a_i$ and block $B_\ell$, use a **rectangle** $(size(a_i), z_i^{(\ell)})$ where $z_i^{(\ell)} = \sum_{b_k \in B_\ell} \sum_{j : a_i \in C_j^{(k)}} \tilde{y}_j^{(k)} > 0$ is the fraction of $a_i$ assigned to $B_\ell$.

A rectangle $(size(a_i), z_i^{(\ell)})$ corresponding to block $B_\ell$ is called

- **wide**, if $size(a_i) > \delta c_{max}^{(\ell)}$.
- **narrow**, if $size(a_i) \leq \delta c_{max}^{(\ell)}$.

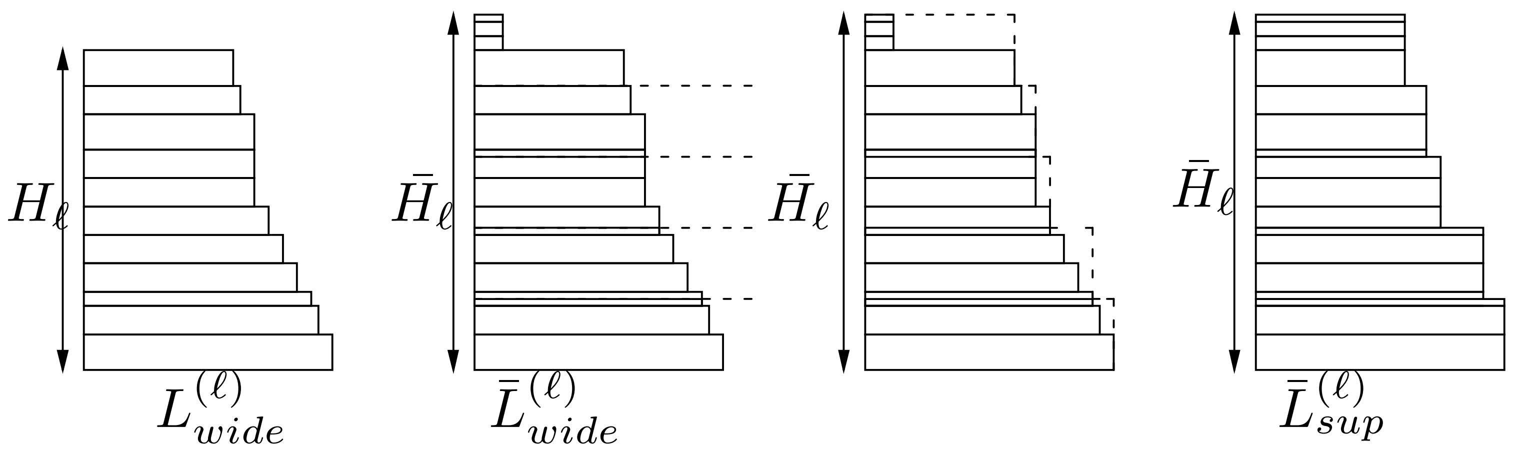
# Rounding the LP-solution III

Build a **stack $St_\ell$ with wide rectangles** ordered by their widths for each block $B_\ell$, $\ell = 1, \dots, t$.



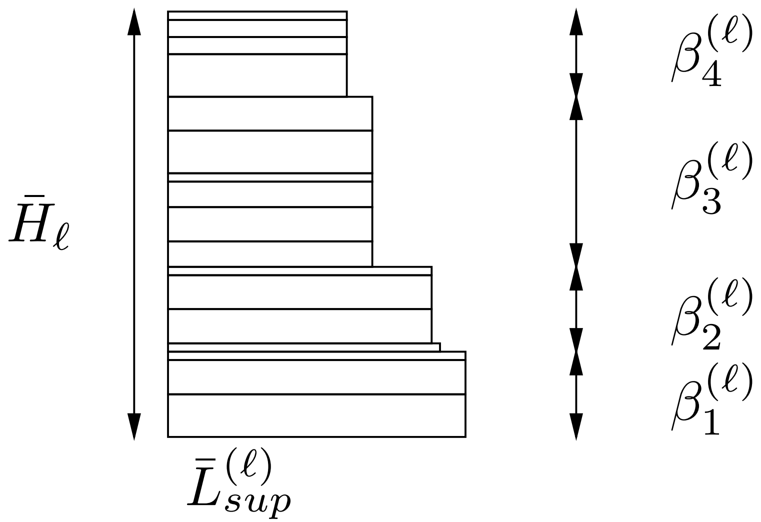
# Rounding the LP-solution IV

- Add **dummy rectangles** with width $\delta^2 c_{max}^{(\ell)}$ until the modified stack has height $\bar{H}_\ell = d_\ell/\delta^2$ where $d_\ell \in \mathbb{Z}^+$.
- Split the modified stack into $1/\delta^2$ groups of height $\delta^2 \bar{H}_\ell = d_\ell$ and **round up the widths** of the rectangles.



# Selecting of items I

- $\bar{L}_{sup}^{(\ell)}$ consists of wide rectangles with a **constant number** $a(\ell) \leq 1/\delta^2$ of different widths $w_1^{(\ell)} > \ldots > w_{a(\ell)}^{(\ell)}$.
- Let $\beta_j^{(\ell)}$ be the total height of rectangles with width $w_j^{(\ell)}$.



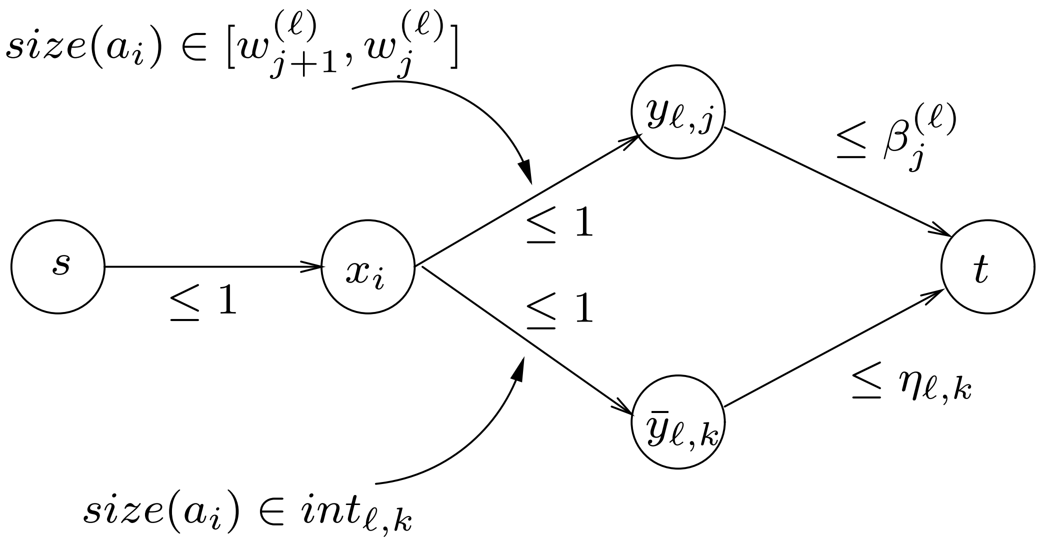
# Selecting of items II

- Let $L^{(\ell)}_{narrow}$ be the set of **narrow rectangles** $(size(a_i), z_i^{(\ell)})$ with $size(a_i) \leq \delta c^{(\ell)}_{max}$ assigned to bins in block $B_\ell$.

- For each interval $int_{\ell,k} = (\frac{\delta}{(1+\delta)^k} c^{(\ell)}_{max}, \frac{\delta}{(1+\delta)^{k-1}} c^{(\ell)}_{max}]$ calculate the **maximum number of items with size in** $int_{\ell,k}$ assigned to block $B_\ell$:

$$\eta_{\ell,k} = \left\lceil \frac{\sum_{i:size(a_i)\in int_{\ell,k}} z_i^{(\ell)} size(a_i)}{\frac{\delta}{(1+\delta)^k} c^{(\ell)}_{max}} \right\rceil .$$

# Network Flow Problem

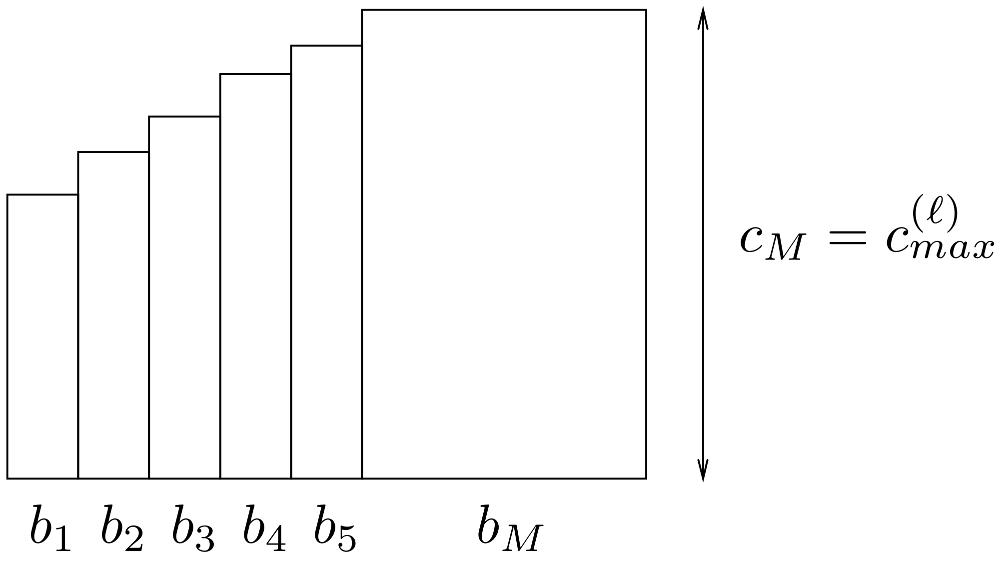



$cost(s, x_i) = profit(a_i)$

Since the capacities are **integral**, we obtain item sets $A^{(\ell)}_{wide}, A^{(\ell)}_{narrow}$ with $profit(\bigcup_\ell A^{(\ell)}_{wide} \cup A^{(\ell)}_{narrow}) \geq (1 - 5\alpha)OPT(LP(A, B))$.

# Packing of selected items

Let $OPT_{ILP}(S, B_\ell)$ be the minimum number of bins of capacity $c_{max}^{(\ell)}$ used for $S$, where the **first $M-1$ bins** are not counted, but can be used for the packing.



# Packing of selected items

**Lemma:**

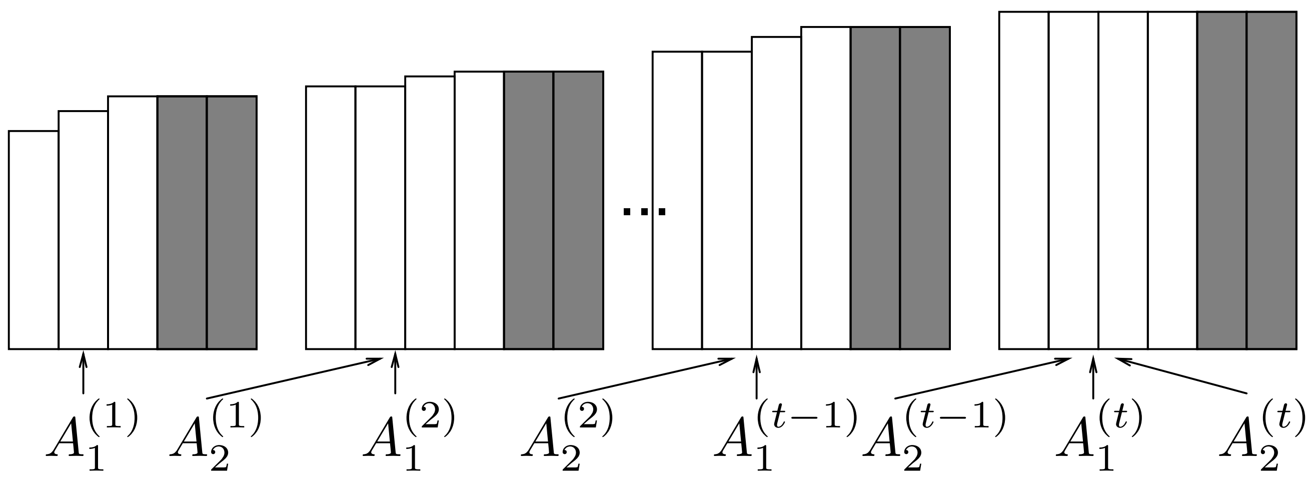
$$OPT_{ILP}(A^{(\ell)}_{wide} \cup A^{(\ell)}_{narrow}) \leq C' \log^2(1/\delta),$$

where $C'$ is a constant.

**Idea of proof:** Calculate **integrality gap** for bin packing with different bin sizes.

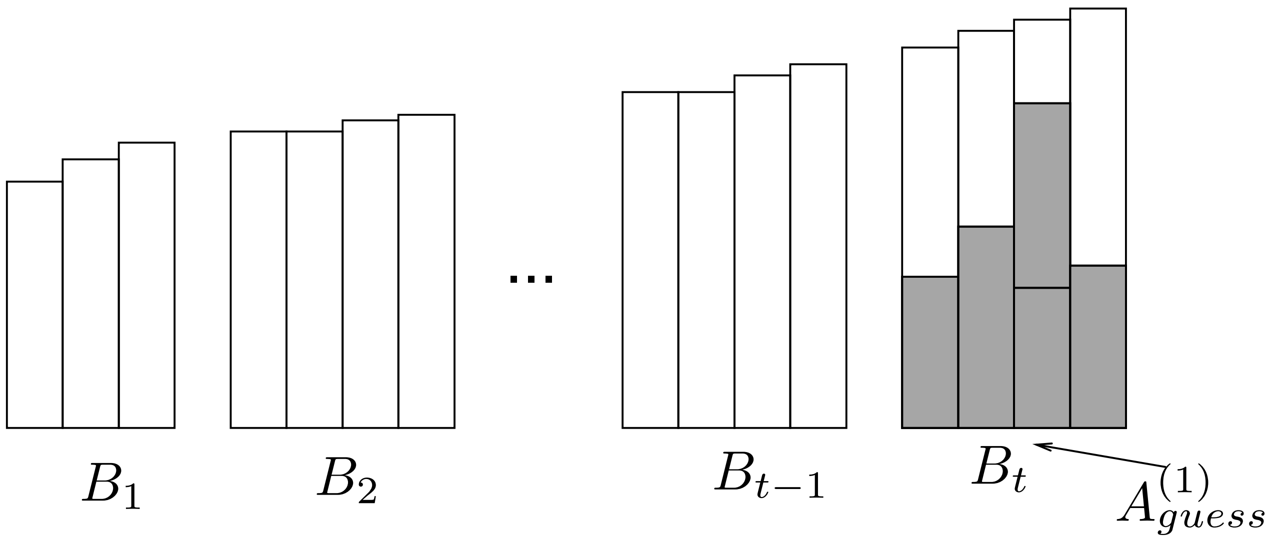
# Shifting argument

- generate a packing for $A^{(\ell)}_{wide} \cup A^{(\ell)}_{narrow}$ into at most $M + \lfloor C' \log^2(1/\delta) \rfloor$ bins.
- a subset $A_1^{(\ell)}$ fits into $B_\ell$ and the remaining set $A_2^{(\ell)}$ fits into $\lfloor C' \log^2(1/\delta) \rfloor$ bins of size $c^{(\ell)}_{max}$.



# General Case: High Profit Items

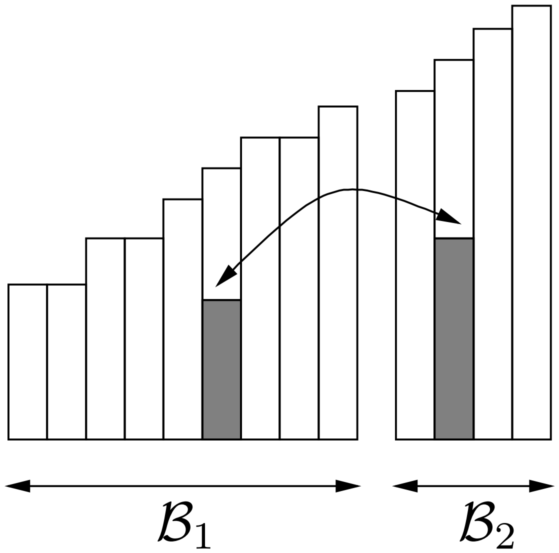
Guess and pack high profit items $A^{(1)}_{guess}$ into the last block $B_t$.




Let $Area_{Rem} = \sum_{b_i \in B_t} c(b_i) - size(A^{(1)}_{guess})$ be the **remaining space** in block $B_t$.

# General Case: Exchange Argument

Use an **exchange argument** for medium sizes between $\mathcal{B}_1 = \cup_{\ell=1}^{t-1} B_\ell$ and $\mathcal{B}_2 = B_t$ such that $\mathcal{B}_2$ contains the larger medium sizes for certain subintervals.



# Advantage of Exchange Step

- Modification generates only one **additional bin** of small size $\delta Area_{Rem}/4$ (that can be eliminated via shifting argument).
- Medium profit items for block $B_t$ can be chosen in $2^{O(1/\delta \log^4(1/\delta))}$ time.
- Small profit items for $\mathcal{B}_2 = B_t$ and other items for $\mathcal{B}_1$ can be chosen via a **modified linear program**.

# Overall Running Time

**Theorem:** The running time of our algorithm is bounded by

$$2^{O(1/\epsilon \log^4(1/\epsilon))} + poly(n).$$

**Note:** Running time can be improved to $2^{O(1/\epsilon \log^2(1/\epsilon))} + poly(n)$ if a **conjecture about the integrality gap for bin packing** is true.

# Related Results

- Scheduling of jobs with non-availability of processors.

  **(a)** $3/2 + \epsilon$ approximation algorithm improving the $(2 + \epsilon)$ approximation algorithm by Scharbrodt, Steger, and Weiser **(Diedrich, Jansen 2009)**.

  **(b)** $3/2$ approximation algorithm with faster running time $O(n \log n + \log(np_{max})(n + T_{MSSP}(n, 1/8)))$ **(Jansen, Prädel, Schwarz, Svensson 2011)**

# Future Work

- faster approximation schemes for the linear program relaxation for MKP.
- faster EPTAS for MKP and small number $m$ of bins.
- faster EPTAS for MKP and special values $\epsilon = 1/4, 1/8$.
- integrality gap between ILP and LP formulations for bin packing.
- lower bound for the running time of the EPTAS.