Counting maximal independent sets in subcubic graphs.

Konstanty Junosza-Szaniawski, Michał Tuczyński

Warsaw University of Technology

SOFSEM 2012

What is known:

Complexity

	Independent sets	Maximal independent sets
$\Delta = 2$	polynomial	polynomial
$\Delta = 3$		
$\Delta = 4$		
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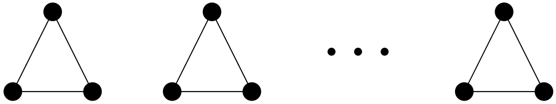
Moon, Moser 1965

Number of MIS is at most $3^{n/3} = 1.44..^n$



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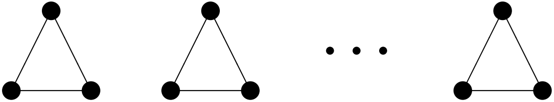

Johnson, Yannakakis 1988

MIS can be generated with polynomial delay

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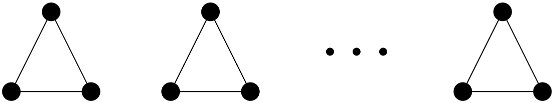

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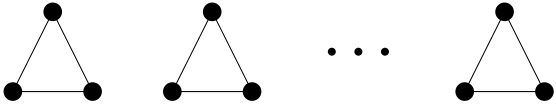

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Gaspers, Kratsch, Liedloff 2008, 2012

Maximal independent sets can be counted in time $O^*(1.3642^n)$
Maximal independent sets in a subcubic graph
can be counted in time $O^*(1.3532^n)$ in polynomial space.

Björklund, Husfeldt 2006

Graph can be colored in time $O^*((1+c)^n)$ and polynomial space, it there is an algorithm counting independent sets in time $O^*(c^n)$ and polynomial space.

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Counting independent sets - not necessary maximal

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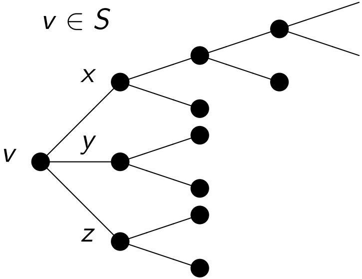
Branching



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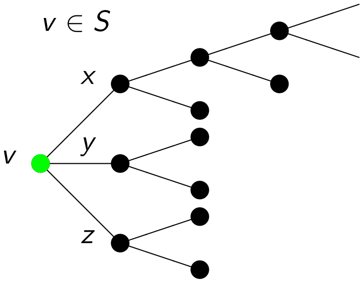

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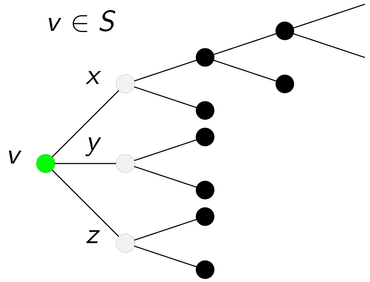

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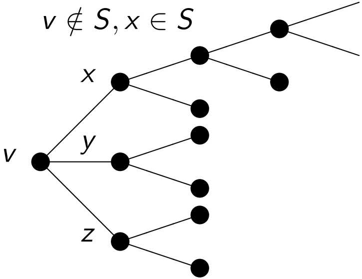

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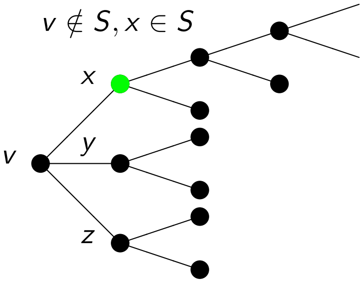

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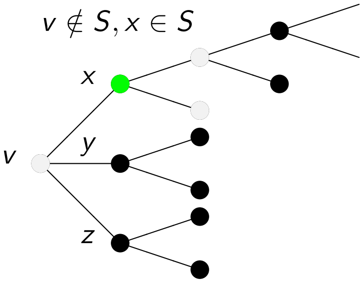

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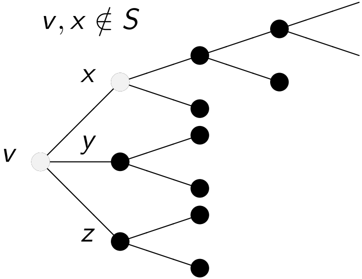

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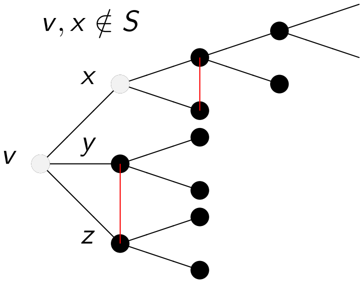
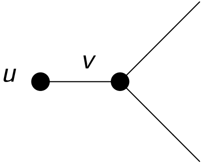


Reduction



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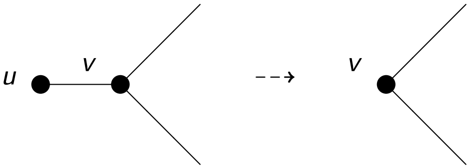

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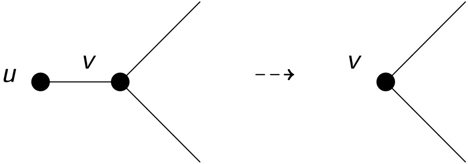
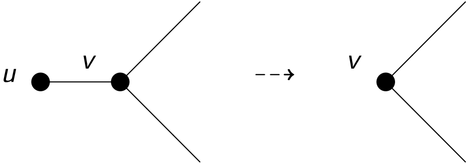


$$c_1(v) := c_1(v) \cdot c_0(u)$$
$$c_0(v) := c_0(v) \cdot (c_1(u) + c_0(u))$$
$$c_{\bar{0}}(v) := c_1(u)$$

$$\sum_{S} \prod_{w:w\in S} \mathbf{c_1}(w) \prod_{\substack{w:w\notin S,\\ \bar{N}(w)\cap S\neq\emptyset}} \mathbf{c_0}(w) \prod_{w:\bar{N}[w]\cap S=\emptyset} \mathbf{c_{\bar{0}}}(w)$$

For $c_1 = 1,\ c_0 = 1,\ c_{\bar{0}} = 0$ the sum is equal to the number of MIS.



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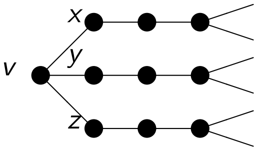
The algorithm

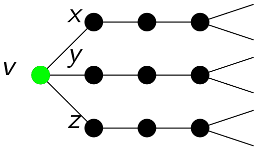
MISCount($G, c_1, c_0, c_{\bar{0}}$)

1. Reduction($G, c_1, c_0, c_{\bar{0}}$)
2. Choose a vertex v
3. Branch on the vertex v

Complexity



$v \in S$



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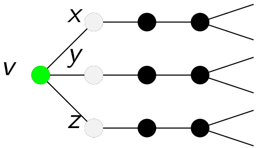

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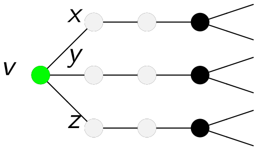

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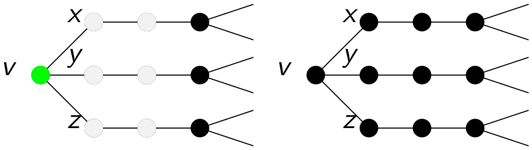

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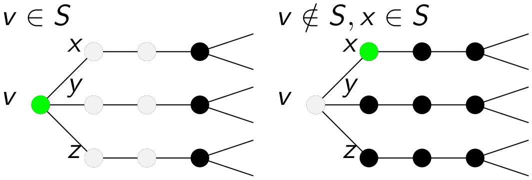

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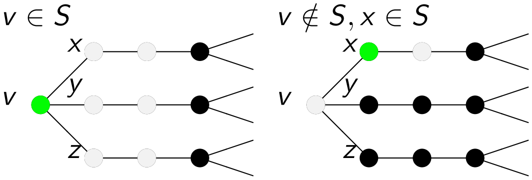

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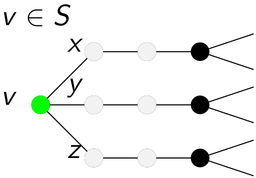

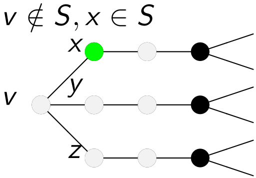

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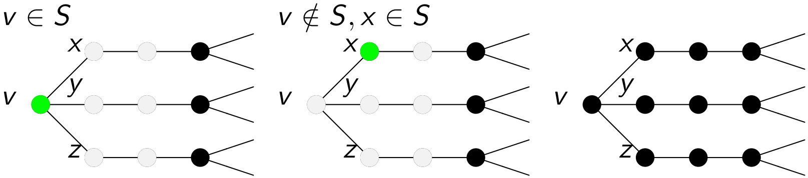

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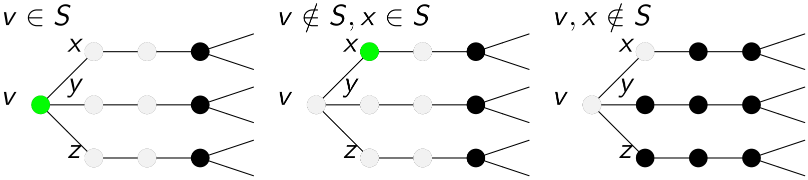

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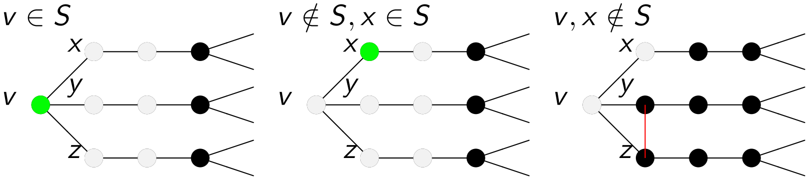

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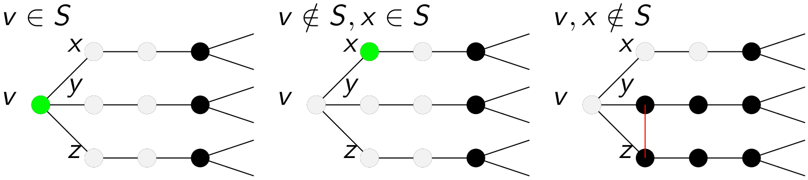

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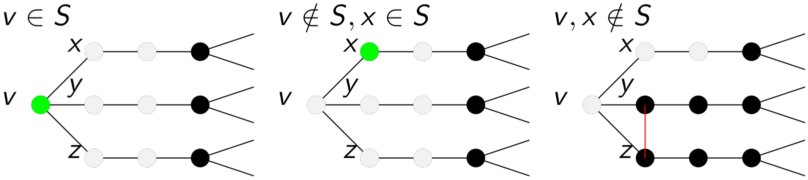

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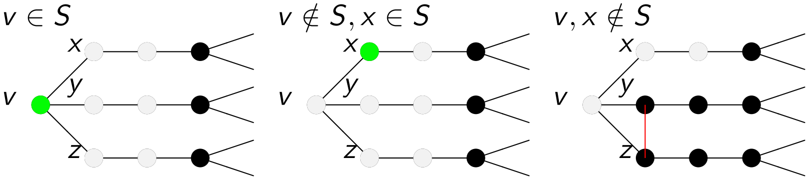

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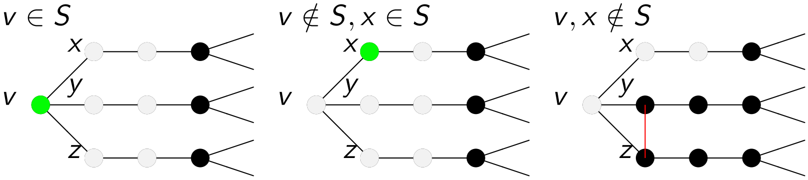

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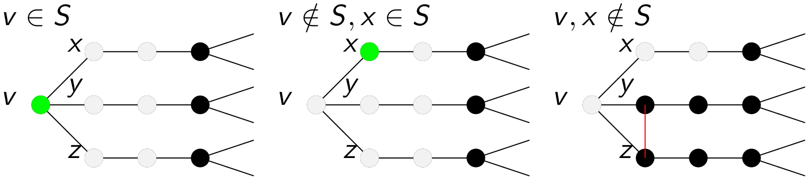

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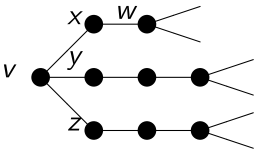

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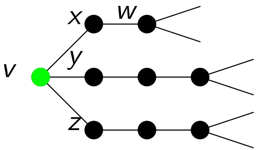

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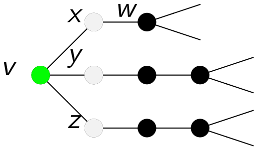

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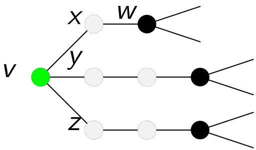

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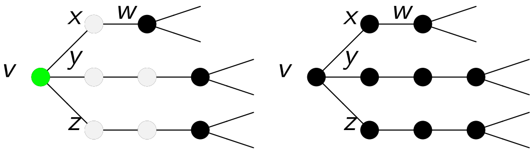

Complexity

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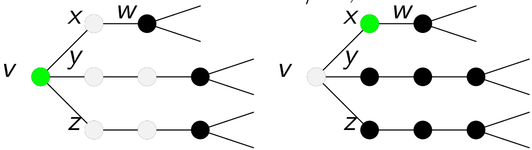
$v \notin S, x \in S$



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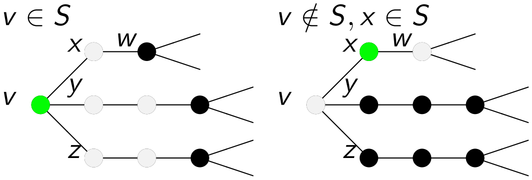

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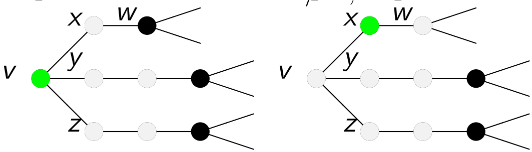

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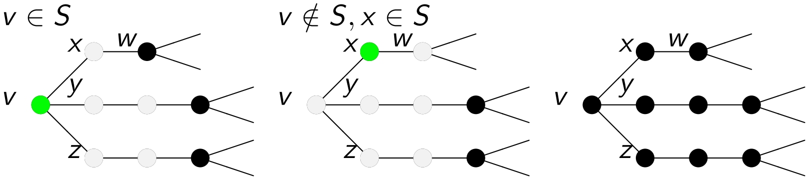

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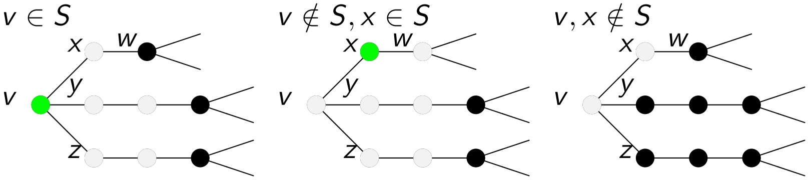

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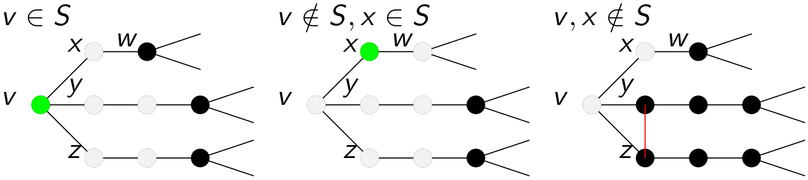

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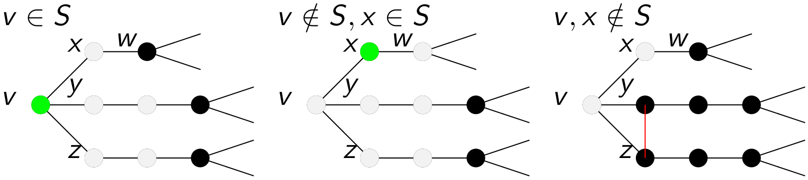

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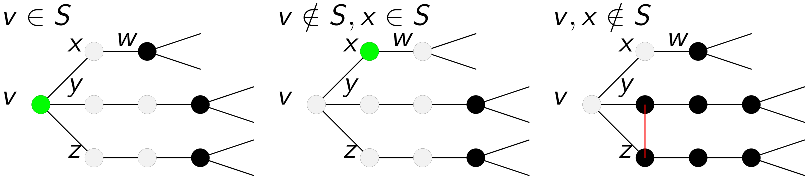

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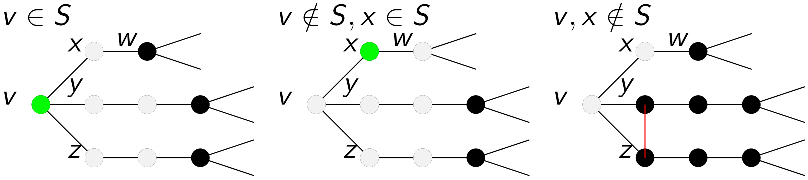
Complexity



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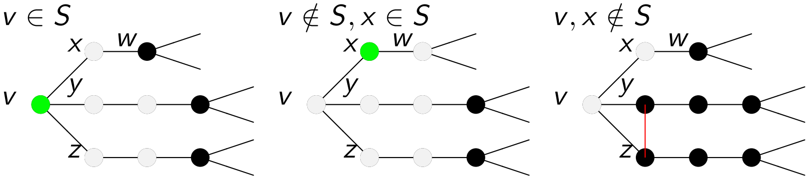

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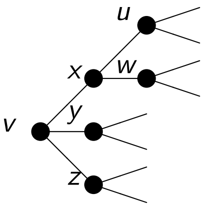

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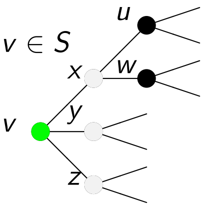

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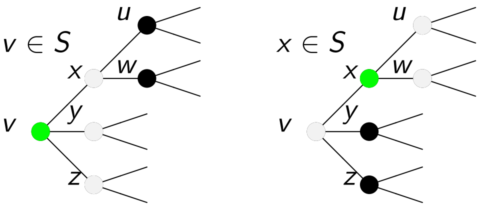

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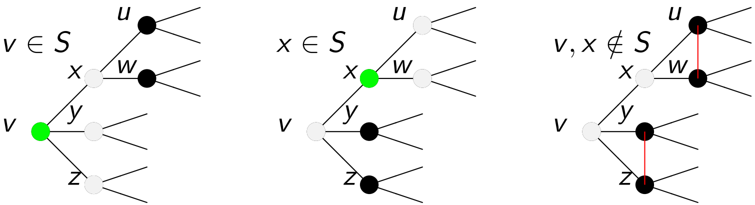

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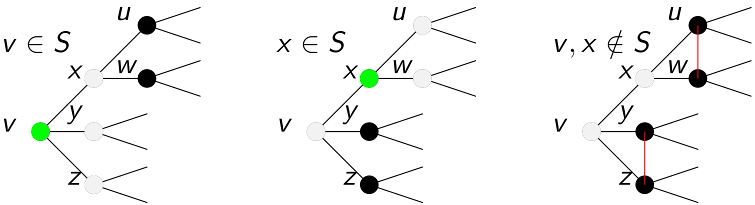

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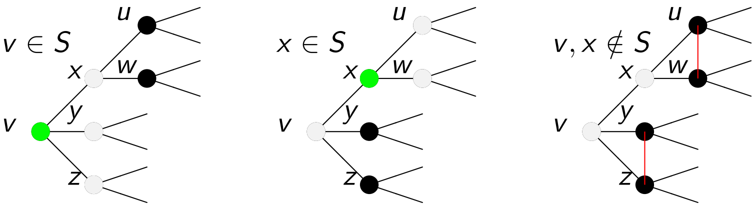
Complexity



$$T(n_3) = T(n_3 - 10) + T(n_3 - 10) + T(n_3 - 2)$$
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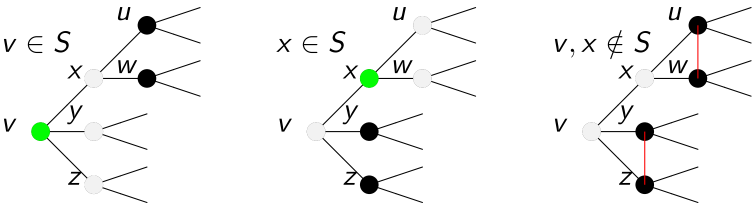
Complexity



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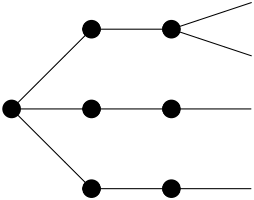
$T(n) = O^*(1.21..^{n})$

Lemma on density

$\frac{2m}{n} > 2.25$



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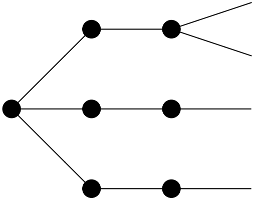
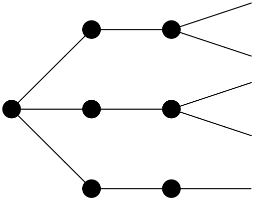
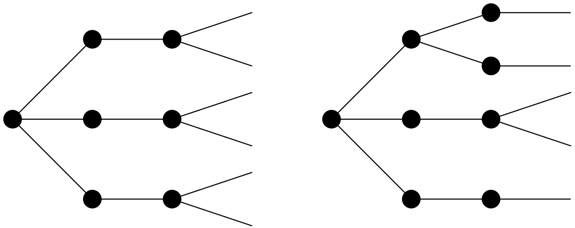


$\frac{2m}{n} > \frac{16}{7}$



Lemma on density

$$\frac{2m}{n} > \frac{7}{3}$$



Lemma on density

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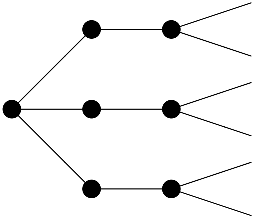
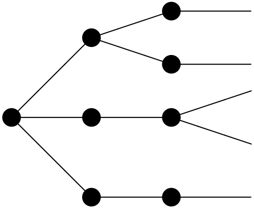
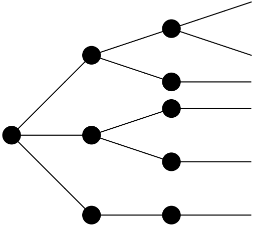




$\frac{2m}{n} > \frac{28}{11}$



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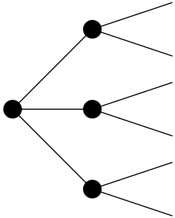

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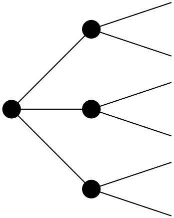
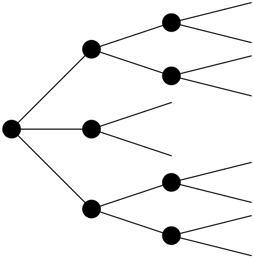


$$\frac{2m}{n} > 2.8$$



Let μ be a measure of a graph, let G_i for $i = 1..t$ be graphs obtained in the i-th branch of recursive call of an algorithm $\mathcal{A}$, let τ_0 be the largest root of

$$\sum_{i=1}^{k} x^{-(\mu(G)-\mu(G_i))} = 1$$

then the algorithm $\mathcal{A}$ runs in time $O^*(\tau_0^{\mu(G)})$.

$$\mu(n_2, n_3) = \begin{cases} \mu_0(n_2, n_3) & \text{if} \quad n = 0 \text{ lub } \frac{2m}{n} = 2 \\ \mu_1(n_2, n_3) & \text{if} \quad 2 < \frac{2m}{n} \le 2\frac{1}{4} \\ \mu_2(n_2, n_3) & \text{if} \quad 2\frac{1}{4} < \frac{2m}{n} \le 2\frac{2}{7} \\ \dots & \\ \mu_7(n_2, n_3) & \text{if} \quad 2\frac{14}{17} < \frac{2m}{n} \le 3. \end{cases}$$

$$\mu_i(n_2, n_3) = w_{i,2} n_2 + w_{i,3} n_3$$

i	p_{i-1}	p_i	$w_{i,2}$	$w_{i,3}$
0		2	0	0
1	2	$2\frac{1}{4}$	0	0.5
2	$2\frac{1}{4}$	$2\frac{2}{7}$	0.018460	0.444619
3	$2\frac{2}{7}$	$2\frac{1}{3}$	0.037065	0.3981058
4	$2\frac{1}{3}$	$2\frac{6}{11}$	0.0520754	0.368086
5	$2\frac{6}{11}$	$2\frac{4}{5}$	0.082954	0.342353
6	$2\frac{4}{5}$	$2\frac{14}{17}$	0.140599	0.327942
7	$2\frac{14}{17}$	3	0.154465	0.324971

Main result

Maximal independent sets in subcubic graph can be counted in time $O^*(1.2526..^n)$

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Future work

- Count minimal models of 2-SAT formula, with variables appearing no more than 3 times.

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THANK YOU