A Combinatorial Algorithm for All-Pairs Shortest Paths Problem in Directed Vertex-Weighted Graphs with Applications to Disc Graphs

Andrzej Lingas Dzmitry Sledneu

Lund University

All-Pairs Shortest Paths Problem

Problem

Given directed weighted graph $G = (V, E, \omega)$, $|V| = n$, $|E| = m$, $\omega : E \to [-\infty, +\infty]$, *compute the shortest distances for all pairs of vertices.*

Known Results

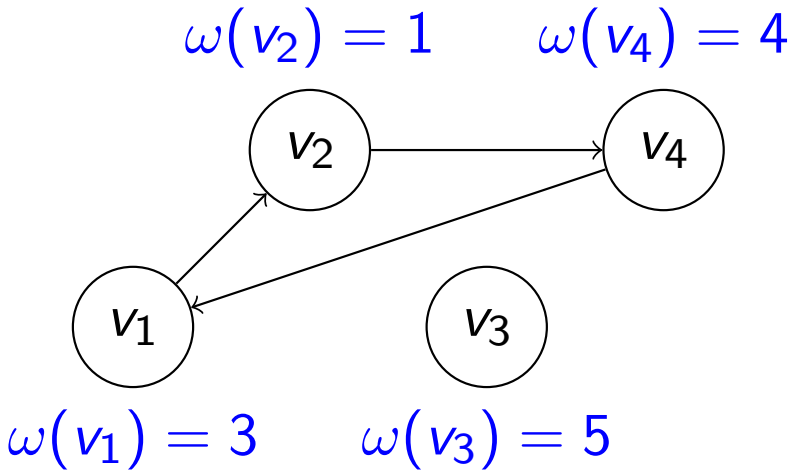
- Floyd, Warshall (1962): $O(n^3)$
- Johnson (1977): $O(nm + n^2 \log n)$
- ...
- Chan (2007): $O(n^3 \log^3 \log n / \log^2 n)$

Is there a truly subcubic ($O(n^{3-\varepsilon})$ time) algorithm?

For some special cases there are.

Vertex-Weighted Graphs

Special case of edge-weighted graphs



$G = (V, E, \omega)$,
$\omega : |V| \to [-\infty, +\infty]$

- Chan (2007): $O(n^{2.844})$
- Yuster (2009): $O(n^{2.842})$

Fast matrix multiplication, huge hidden constant.

Is there a truly subcubic combinatorial algorithm?

For some special graphs there are.

Highly Clustered Graphs

A – adjacency matrix of G.
$H(A)$ – complete weighted graph on the rows of A,
$\omega(row_i, row_j) =$ Hamming distance between row_i and row_j
(n-dimensional metric space).

Definition

G is highly clustered if $MST(H(A)) = O(n^{2-\varepsilon})$,
where $MST(H(A))$ is a cost of the min spanning tree for $H(A)$.

Fact (Indyk (2000))

A $\Theta(\log n)$-approximate minimum spanning tree for $H(A)$ can be computed by a Monte Carlo algorithm in time $\tilde{O}(n^2)$.

Related Problem: Boolean Matrix Multiplication

Special case of APSP – undirected, unweighted graph.

- Coopersmith, Winograd (1990): $O(n^{2.376})$
  (fast matrix multiplication)
- Bansal, Williams (2009): $O(n^3 \log^2 \log n / log^{9/4} n)$
  (combinatorial)
- Björklund, Lingas (2001): $\tilde{O}(cn + n^2)$,
  where $c = \min\{MST(H(A)), MST(H(B^T))\}$
  (combinatorial, truly subcubic for highly clustered graphs)

Björklund et al. algorithm cannot be used for transitive closure, since highly clusterness goes away in power graphs.

Mixed Matrix Products

Definition

Matrix $C \in M_n([-\infty, +\infty])$ is a mixed right product of $A \in M_n([-\infty, +\infty])$ and $B \in M_n(\{0, 1\})$,

$$\text{if } C[i,j] = \begin{cases} \min\limits_{1 \le k \le n} \{A[i,k] \mid B[k,j] = 1\}, & \exists k \text{ s.t. } B[k,j] = 1 \\ +\infty, & \textit{otherwise}. \end{cases}$$

Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} +\infty & 1 \\ +\infty & 3 \end{pmatrix}$$

Mixed Matrix Products

Definition

Matrix $C \in M_n([-\infty, +\infty])$ is a mixed left product of $B \in M_n(\{0,1\})$ and $A \in M_n([-\infty, +\infty])$,

$$\text{if } C[i,j] = \begin{cases} \min\limits_{1 \le k \le n} \{A[k,j] \mid B[i,k] = 1\}, & \exists k \text{ s.t. } B[i,k] = 1 \\ +\infty, & \textit{otherwise}. \end{cases}$$

Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}$$

Fast Computation of Right Mixed Matrix Product

1. Compute the $O(\log n)$-approximate MST of $H(B^T)$.
   Fix some traversal $(col_1, col_2, \ldots, col_n)$ of the tree.
2. For $j = 1, \ldots, n$ compute

$$Ins[col_j] = \begin{cases} \{k \mid B[k, col_j] = 1\}, & k = 1, \\ \{k \mid B[k, col_j] - B[k, col_{j-1}] = 1\}, & k > 1 \end{cases}$$

$$Del[col_j] = \begin{cases} \varnothing, & k = 1, \\ \{k \mid B[k, col_{j-1}] - B[k, col_j] = 1\}, & k > 1 \end{cases}$$

Note: $|Del[col_j]| + |Ins[col_j]|$ equals to the Hamming distance between col_j and col_{j-1}.

Fast Computation of Right Mixed Product

3. For $row = 1, \ldots, n$

   3.1 $Heap.Init()$

   3.2 For $j = 1, \ldots, n$

   For all $k \in Ins[col_j]$ do $Heap.Insert(A[i, k])$

   For all $k \in Del[col_j]$ do $Heap.Delete(A[i, k])$

$$C[row, col_j] = \begin{cases} +\infty, & Heap.IsEmpty() = \textbf{true}, \\ Heap.Min(), & otherwise. \end{cases}$$

Running Time

Lemma

Algorithm runs in time $\tilde{O}(n^2 + nMST(H(B^T)))$.

Proof.

Step 1 (MST computation) – $\tilde{O}(n^2)$.
Step 2 (*Ins*, *Del* computation) – $O(n^2)$.
Step 3 (product computation) – $\tilde{O}(nMST(H(B^T)))$.

□

APSP for Vertex-Weighted Graphs

$G = (V, E, \omega), |V| = n, \omega : V \to [0, +\infty]$, $A \in M_n(\{0, 1\})$ – adjacency matrix.

Let $D \in M_n([0, +\infty])$, where $D(v, u)$ is a shortest distance between v and u.

Let $D^{(i)} \in M_n([0, +\infty])$, where $D^{(i)}(v, u)$ is a shortest distance between v and u on paths of at most i edges.

$$D^{(0)}(v, u) = \begin{cases} 0, & v = u, \\ +\infty, & \textit{otherwise}, \end{cases}$$

$$D^{(1)}(v, u) = \begin{cases} \omega(v) + \omega(u), & (v, u) \in E, \\ +\infty, & \textit{otherwise} \end{cases}$$

Reduction to Mixed Matrix Product

Lemma

$D^{(i+1)}$ can be computed from $D^{(i)}$ and right mixed product of $D^{(i)}$ and A (left mixed product of A and $D^{(i)}$) in time $O(n^2)$.

Proof.

$$D^{(i+1)}[v,u] = \min\{\{D^{(i)}[v,u]\} \cup \{\min_{1 \le k \le n}\{D^{(i)}[v,k] + \omega(u) | A[k,u] = 1\}\}\}$$

□

Hitting Set

A set Q – of all q-vertex shortest paths can be constructed by backtracking from $D^{(0)}, \ldots, D^{(q-1)}$ in time $O(qn^2)$.

Fact (Hitting set greedy heuristic)

Given Q – a collection of $O(n^2)$ paths, where each paths consists of exactly q vertices, we can find a subset B_Q of size $O((n/q)\log n)$ called bridging set that hits all paths in Q and we can do it in time $O(qn^2)$.

Algorithm for APSP

1. Compute $D^{(0)}, \ldots, D^{(q-1)}$.
2. Compute Q – a set of all q-vertex shortest paths.
3. Compute B_Q – $O((n/q)\log n)$ bridging set for Q.
4. Run Dijkstra's algorithm for all vertices from B_Q.
5. Run Dijkstra's algorithm for all from B_Q in reversed graph.
6. For all pairs of vertices $(v, u), v \notin B_Q, u \notin B_Q$ set

$$D[v,u] = \min\{D^{(q-1)}[v,u], \min_{k \in B_Q}\{D[v,k] + D[k,u] - \omega(k)\}\}.$$

Running Time

Theorem

All-pairs shortest paths for a directed graph with nonnegative vertex weights can be computed by a combinatorial randomized algorithm in time $\tilde{O}(n^2\sqrt{n+c})$, *where* $c = \min\{MST(H(A)), MST(H(A^T))\}$.

Proof.

Step 1 ($D^{(0)}, \ldots, D^{(q-1)}$ computation) – $\tilde{O}(q(cn+n^2))$.
Steps 2-3 (Q and B_Q computation) – $O(qn^2)$.
Steps 4-5 (Dijkstra's algorithm) – $\tilde{O}((n/q)n^2)$.
Step 6 – $\tilde{O}((n/q)n^2)$.
Overall – $\tilde{O}(q(cn+n^2))$, set $q = \sqrt{\frac{n^3}{cn+n^2}}$.

Uniform Disk Graphs of Bounded Density

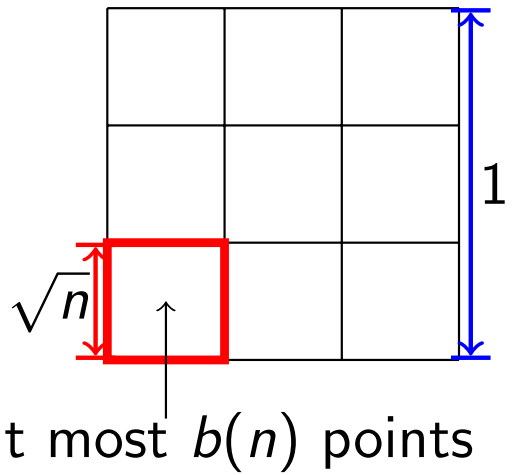


Figure: $b(n)$-dense point set

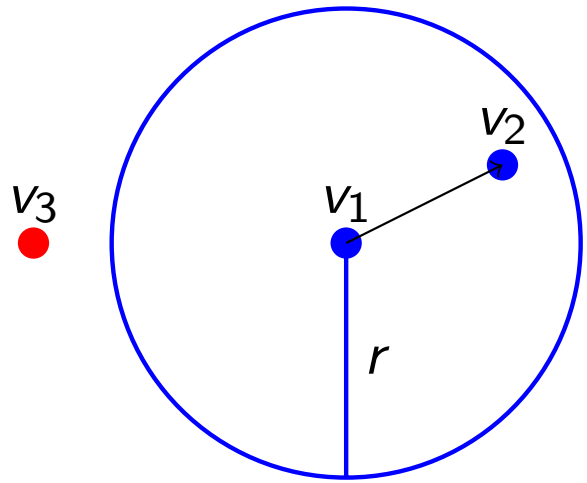


Figure: $(v_1, v_2) \in E, (v_1, v_3) \notin E$

APSP for Vertex-Weighted Uniform Disk Graphs

Theorem

The APSP problem for a vertex-weighted uniform disk graph induced by a $b(n)$-dense point set can be solved by a combinatorial algorithm in time

$$\tilde{O}(\sqrt{r}n^{2.75}\sqrt{b(n)}).$$

Corollary

For vertex-weighted uniform disk graphs of unit radius induced by $O(1)$-dense point set APSP can be solved in time

$$\tilde{O}(n^{2.75}).$$

Open Questions

- More graph classes with property

$$c = \min\{MST(H(A)), MST(H(A^T))\} = O(n^{3-\varepsilon}).$$

- Generalization for edge-weighted graphs.

Thank You!

Questions?