

# Complexity of Model Checking for Modal Dependence Logic

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# Overview

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- Kripke Structure
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## 3 Complexity Results on Model Checking for MDL

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# Motivation

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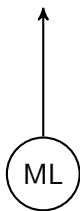
- Dependence atom extends ordinary modal logic
- CTL is used for verifying systems e.g. integrated circuits
- MDL seen as first approach of DCTL (= CTL + dependence atom)
- MDL is based on teams, not on single assignments / worlds

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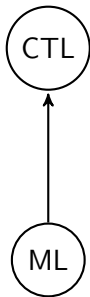




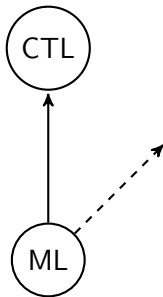
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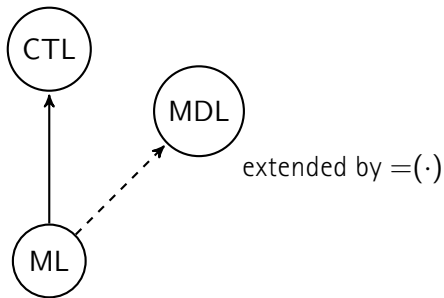
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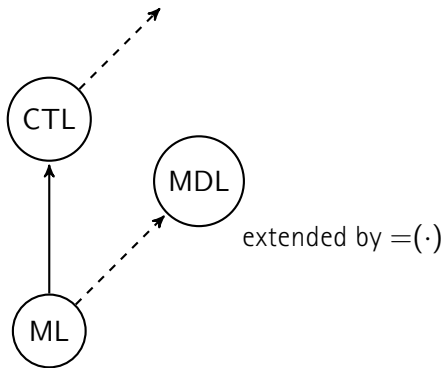
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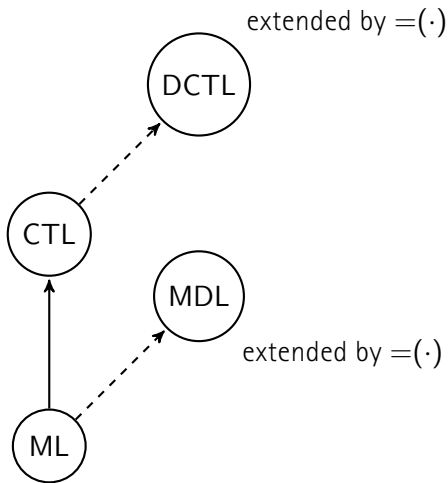
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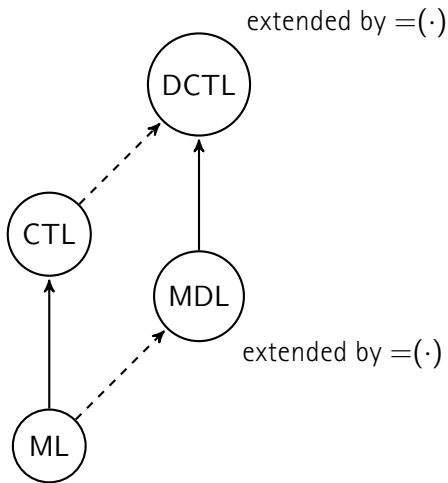
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If  $s \in S$ ,  $s$  is called **world**. We call  $S$  the **set of worlds**.  $R$  is called the **transition relation** between the worlds, and  $\pi$  is called the **labeling function**.

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 \varphi ::= & \mathbf{q} \mid \neg \mathbf{q} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Box \varphi \mid \Diamond \varphi \mid \mathbf{=(p_1, \dots, p_n; q)} \mid \\
 & \mathbf{\neg=(p_1, \dots, p_n; q)}
 \end{aligned}$$

# Modal Logic

## Semantics of Modal Logic

Let  $W = (S, R, \pi)$  an *AP*-Kripke structure,  $T \subseteq S$  a set of initial states, and  $\varphi$  an ML formula and  $p \in AP$ . Inductive over the structure of  $\varphi$  we define

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 W, T \models \varphi \vee \psi & \text{iff there are sets } T_1, T_2 \subseteq T \text{ with} \\
 & T_1 \cup T_2 = T, \\
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$W, T \models \varphi \wedge \psi$	iff	$W, T \models \varphi$ and $W, T \models \psi$

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# Modal Dependence Logic

## Semantics of the Dependence Atom

Let  $W = (S, R, \pi)$  an *AP*-Kripke structure,  $T \subseteq S$  the evaluation team. Let  $p_1, \dots, p_n, q \in AP$ . Then  $\models(p_1, \dots, p_n; q)$  is called a **dependence atom**. The **truth** of the dependence atom on  $W, T$  is defined by

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$$W, T \models \models(p_1, \dots, p_n; q) \quad \text{iff} \quad \text{for all } s_1, s_2 \in T \text{ it holds that} \\
 \pi(s_1) \cap \{p_1, \dots, p_n\} \neq \pi(s_2) \cap \{p_1, \dots, p_n\} \\
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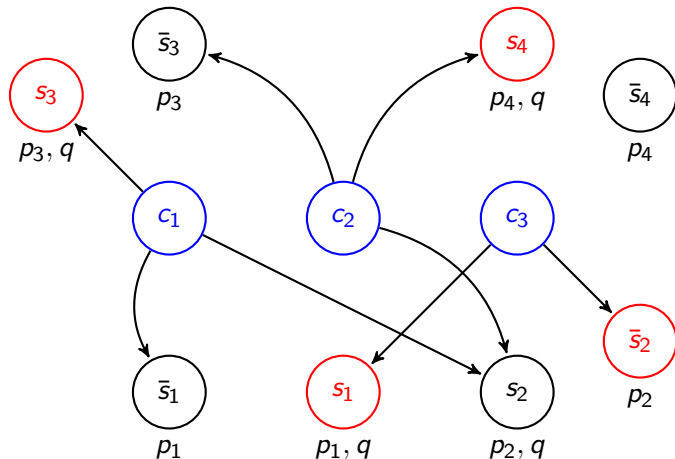
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 W, T \models \neg \models(p_1, \dots, p_n; q) & \quad \text{iff} \quad T = \emptyset
 \end{aligned}$$

# Modal Dependence Logic

Example: Let  $\varphi := \diamond = (p_1, \dots, p_4; q)$  and  $T := \{c_1, c_2, c_3\}$ .

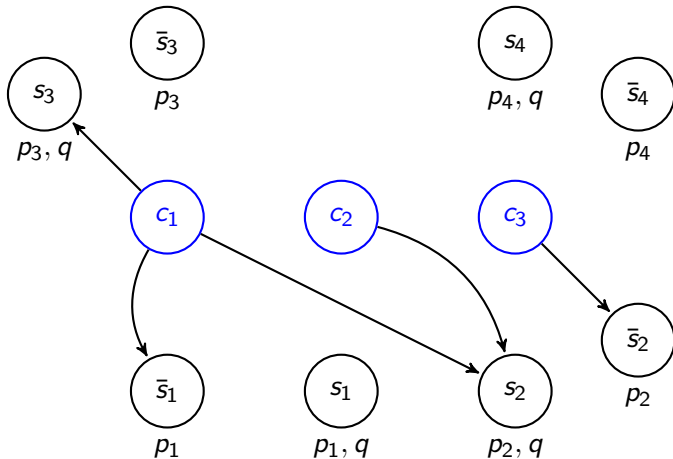
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## The Model Checking problem

Let  $M \subseteq \{\Box, \Diamond, \wedge, \vee, \neg, =\}$ .

Given: A Kripke structure  $W = (S, R, \pi)$ , a formula  $\varphi \in \text{MDL}(M)$  and an evaluation team  $T \subseteq S$ .

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$$\text{MDL-MC}(M) := \left\{ \langle W, T, \varphi \rangle \mid \begin{array}{l} W = (S, R, \pi) \text{ an AP-Kripke structure,} \\ T \subseteq S, \varphi \in \text{MDL}(M) \text{ and } W, T \models \varphi \end{array} \right\}.$$

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Model checking for modal dependence logic is NP-complete.

We have to show:

- MDL-MC is in NP
- MDL-MC is NP-hard



# The MDL Model Checking problem is in NP

## Algorithm

```

bool check( $W = (S, R, \pi)$ ,  $\varphi$ ,  $T$ )
  case  $\varphi$ 
  when  $\varphi = p$ 
    foreach  $s_i \in T$ 
      if not  $p \in \pi(s_i)$  then
        return false;
    return true;
  when  $\varphi = \neg p$ 
    foreach  $s_i \in T$ 
      if  $p \in \pi(s_i)$  then
        return false;
    return true;
  
```

# The MDL Model Checking problem is in NP

## Algorithm

```

when  $\varphi = \psi \vee \theta$ 
  guess two sets  $A, B \subseteq T$ ;
  if not  $A \cup B = T$  then
    return false;
  return (check ( $W, \psi, A$ ) and check ( $W, \theta, B$ ));
when  $\varphi = \psi \wedge \theta$ 
  return (check ( $W, \psi, T$ ) and check ( $W, \theta, T$ ));
  
```

# The MDL Model Checking problem is in NP

## Algorithm

```

when  $\varphi = \diamond \psi$ 
  guess set of states  $T'$  ;
  foreach  $s \in T$ 
    if there is no  $s' \in T'$  with  $(s, s') \in R$  then
      return false;
  return check  $(W, \psi, T')$ ;
when  $\varphi = \square \psi$ 
   $T' := \emptyset$ ;
  foreach  $s' \in S$ 
    foreach  $s \in T$ 
      if  $(s, s') \in R$  then
         $T' \leftarrow T' \cup \{s'\}$ 
  return check  $(W, \psi, T')$ ;
  
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## Algorithm

```

when  $\varphi = \models (p_1, \dots, p_n; q)$ 
  foreach  $(s, s') \in T \times T$ 
    if  $\pi(s) \cap \{p_1, \dots, p_n\} = \pi(s') \cap \{p_1, \dots, p_n\}$  then
      if not  $\pi(s) \cap \{q\} = \pi(s') \cap \{q\}$ 
        return false;
  return true;

when  $\varphi = \neg \models (p_1, \dots, p_n; q)$ 
  return  $T = \emptyset$ 
  
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## Proposition

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MDL-MC is NP-hard.

We reduce **3SAT** onto MDL-MC, i.e. for every 3SAT formula  $\phi$ , we find an instance of MDL-MC which is at least as hard as  $\phi$ .

# Complexity Results

Operators					Complexity
$\vee$	$\wedge$	$\square$	$\diamond$	$=(\cdot)$	
+	+	*	*	+	NP-complete
*	*	*	+	+	NP-complete
+	*	+	*	+	NP-complete
-	*	*	-	*	in P
+	-	-	-	+	in NP

## Complexity Results with bounded $=(\cdot)$

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$\vee$	$\wedge$	$\square$	$\diamond$	$=(\cdot)$	
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*	+	*	+	+	NP-complete
+	*	+	*	+	NP-complete
-	-	*	*	*	in P
-	*	*	-	*	in P
*	-	-	-	*	in P

We limit the arity of the dependence atom, i.e. all dependence atoms are of the form  $=(p_1, \dots, p_j; q)$  where  $j < k$  with a fixed  $k \in \mathbb{N}$ .



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Thank you!