

Iterated Hairpin Completions of Non-crossing Words

Lila Kari¹ Steffen Kopecki¹ Shinnosuke Seki²

¹Department of Computer Science
The University of Western Ontario

²Department of Systems Bioscience for Drug Discovery,
Kyoto University

SOFSEM 2012



DNA Biochemistry

DNA Strand

Sequence of nukleobases

Adenine, Cytosine, Guanine, Thymine

5'-C-G-G-T-A-T-C-A-T-C-C-C-A-3'

DNA Biochemistry

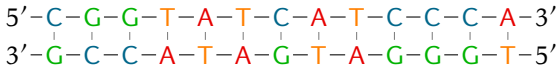
DNA Strand

Sequence of nukleobases

Adenine, Cytosine, Guanine, Thymine

Watson-Crick complement

$\bar{A} = T$ and $\bar{C} = G$



DNA Biochemistry

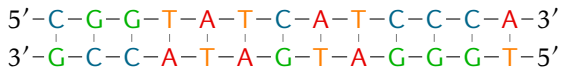
DNA Strand

Sequence of nukleobases

Adenine, Cytosine, Guanine, Thymine

Watson-Crick complement

$\bar{A} = T$ and $\bar{C} = G$



Formal Languages

Alphabet Σ

Complement $\bar{\cdot} : \Sigma \rightarrow \Sigma$

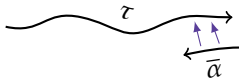
$$\overline{a_1 \cdots a_m} = \overline{a_m} \cdots \overline{a_1}$$

Polymerase Chain Reaction (PCR)



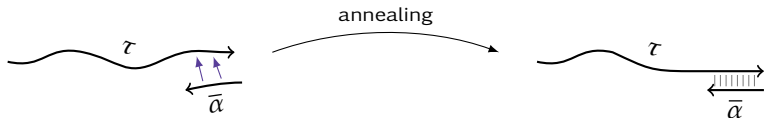
Template $\tau = \sigma\alpha$

Polymerase Chain Reaction (PCR)



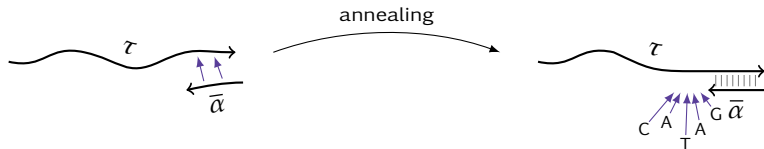
Template $\tau = \sigma\alpha$
Primer $\bar{\alpha}$

Polymerase Chain Reaction (PCR)



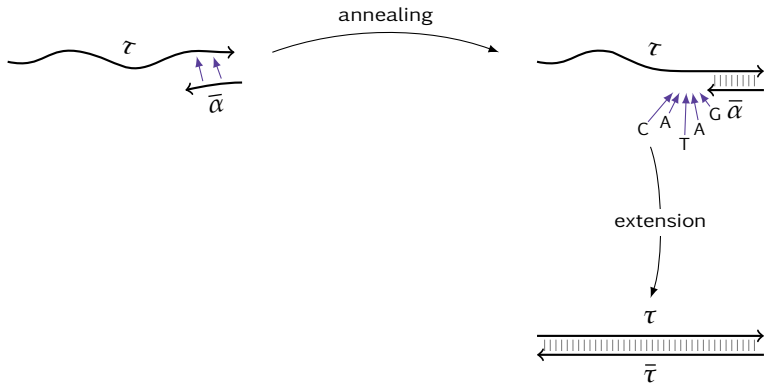
Template $\tau = \sigma\alpha$
Primer $\bar{\alpha}$

Polymerase Chain Reaction (PCR)



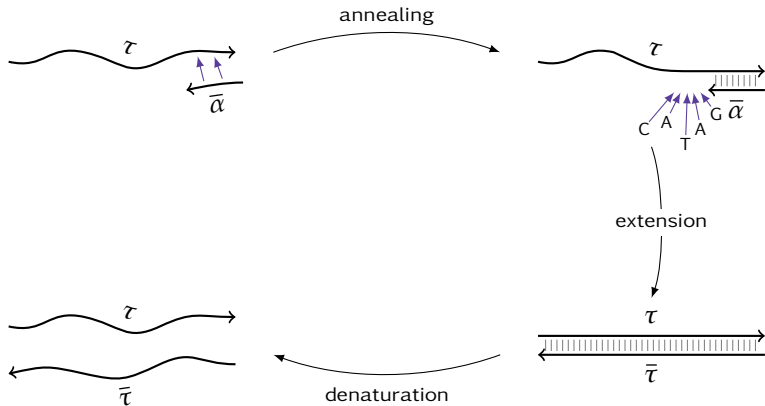
Template $\tau = \sigma\alpha$
Primer $\bar{\alpha}$

Polymerase Chain Reaction (PCR)



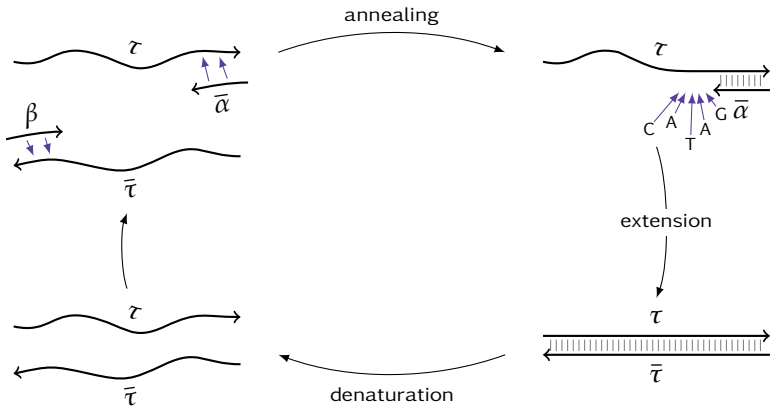
Template $\tau = \sigma\alpha$
Primer $\bar{\alpha}$

Polymerase Chain Reaction (PCR)



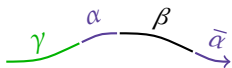
Template $\tau = \sigma\alpha$
 Primer $\bar{\alpha}$

Polymerase Chain Reaction (PCR)

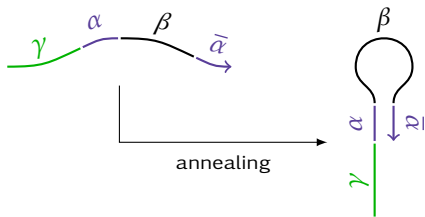


Template $\tau = \sigma\alpha = \beta\sigma'$
 Primer $\bar{\alpha}, \beta$

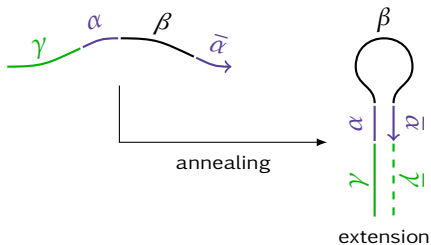
Hairpin Completion



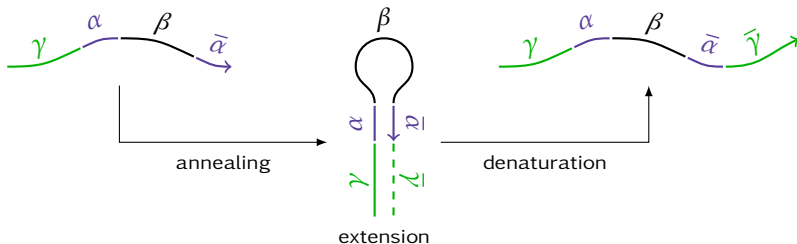
Hairpin Completion



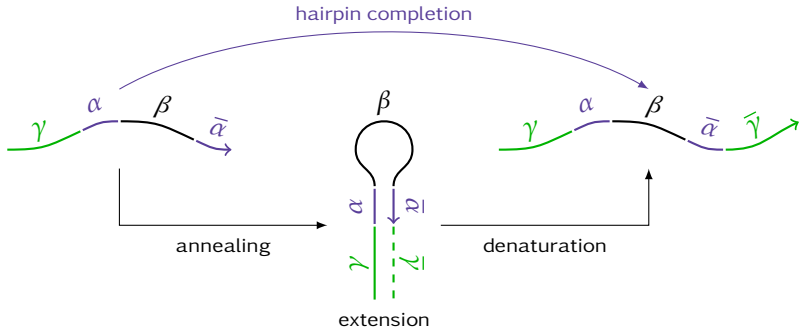
Hairpin Completion



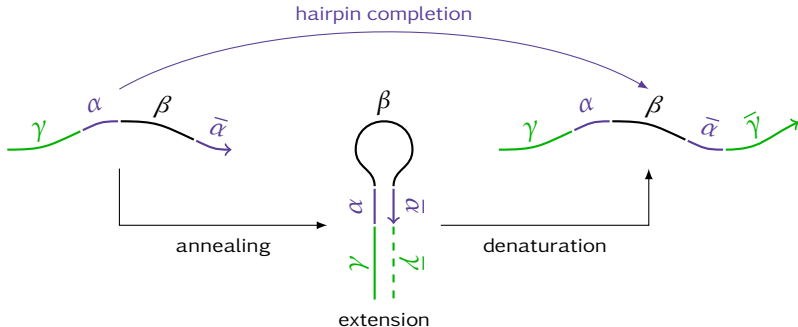
Hairpin Completion



Hairpin Completion



Hairpin Completion



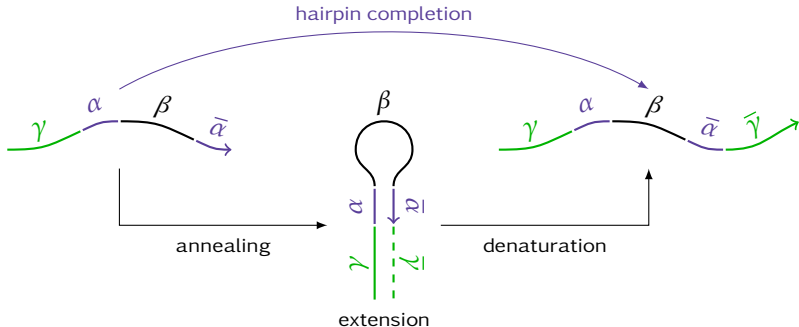
Primer: α

Hairpin completion:

$$\gamma\alpha\beta\bar{\alpha} \rightarrow_{\mathcal{R}} \gamma\alpha\beta\bar{\alpha}\bar{\gamma}$$

$$\alpha\beta\bar{\alpha}\bar{\gamma} \rightarrow_{\mathcal{L}} \gamma\alpha\beta\bar{\alpha}\bar{\gamma}$$

Hairpin Completion



Primer: α

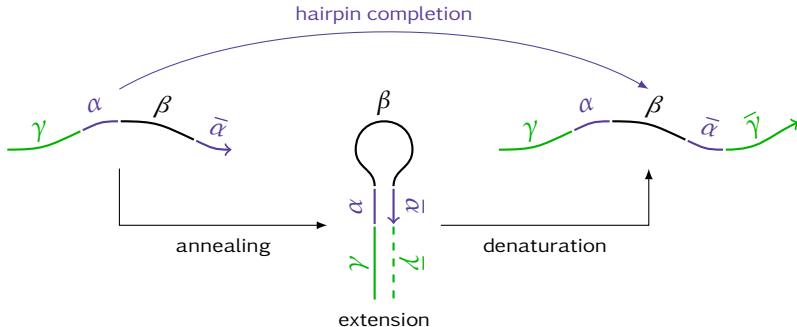
Hairpin completion:

$$\gamma\alpha\beta\bar{\alpha} \rightarrow_{\mathcal{R}} \gamma\alpha\beta\bar{\alpha}\bar{\gamma}$$

$$\alpha\beta\bar{\alpha}\bar{\gamma} \rightarrow_{\mathcal{L}} \gamma\alpha\beta\bar{\alpha}\bar{\gamma}$$

$$w \rightarrow z \quad \text{if } w \rightarrow_{\mathcal{R}} z \text{ or } w \rightarrow_{\mathcal{L}} z$$

Hairpin Completion



Primer: α

Hairpin completion:

$$\gamma\alpha\beta\bar{\alpha} \rightarrow_{\mathcal{R}} \gamma\alpha\beta\bar{\alpha}\bar{\gamma}$$

$$\alpha\beta\bar{\alpha}\bar{\gamma} \rightarrow_{\mathcal{L}} \gamma\alpha\beta\bar{\alpha}\bar{\gamma}$$

$$w \rightarrow z \quad \text{if } w \rightarrow_{\mathcal{R}} z \text{ or } w \rightarrow_{\mathcal{L}} z$$

Iterated hairpin completion:


$$\mathcal{H}^*(w) = \{z \mid w \rightarrow^* z\}$$

Example

$ab\alpha c\alpha\bar{\alpha}\bar{d}\bar{\alpha}\bar{d}\bar{\alpha}$


Example

$ab\alpha c\alpha\bar{a}\bar{d}\bar{a}\bar{d}\bar{a}\bar{b}\bar{a}$




Example

$ab\alpha c\alpha\bar{\alpha}d\bar{\alpha}d\bar{\alpha}(\bar{b}\bar{\alpha})^i$

A diagram showing the word $ab\alpha c\alpha\bar{\alpha}d\bar{\alpha}d\bar{\alpha}(\bar{b}\bar{\alpha})^i$. A horizontal bracket is drawn under the first part of the word, starting from the first 'a' and ending under the second 'd'. An arrow points from the right end of this bracket to the exponent 'i' in the final part of the word.


Example

$\alpha b \alpha c \alpha \bar{\alpha} \bar{d} \bar{\alpha} \bar{d} \bar{\alpha} (\bar{b} \bar{\alpha})^i \bar{c} \bar{\alpha} \bar{b} \bar{\alpha}$

A purple horizontal bracket is drawn under the first part of the word, starting from the first 'α' and ending under the first 'α' of the final 'ᾱbᾱ' segment. A purple arrow starts from the right end of this bracket and points to the right, ending under the final 'ᾱ' of the word.


Example

$ab\alpha c(\alpha b)^i \alpha d\alpha b\alpha c\alpha \bar{\alpha} \bar{d} \bar{\alpha} \bar{d} \bar{\alpha} \underline{(\bar{b}\bar{\alpha})^i \bar{c}\bar{\alpha}\bar{b}\bar{\alpha}}$



Example

$\alpha b \alpha c (\alpha b)^i \alpha d \alpha b \alpha c (\alpha b)^i \alpha d \alpha b \alpha c \alpha \bar{\alpha} \bar{d} \bar{\alpha} \bar{d} \bar{\alpha} (\bar{b} \bar{\alpha})^i \bar{c} \bar{\alpha} \bar{b} \bar{\alpha}$



Example

$\alpha b \alpha c (\alpha b)^i \alpha d \alpha b \alpha c (\alpha b)^i \alpha d \alpha b \alpha c \alpha \bar{\alpha} \bar{d} \bar{\alpha} \bar{d} \bar{\alpha} (\bar{b} \bar{\alpha})^i \bar{c} \bar{\alpha} \bar{b} \bar{\alpha}$

Example

$$\alpha b \alpha c (\alpha b)^i \alpha d \alpha b \alpha c (\alpha b)^i \alpha d \underbrace{\alpha b \alpha c \alpha \bar{\alpha} \bar{d} \bar{\alpha} \bar{d} \bar{\alpha}}_{= z} (\bar{b} \bar{\alpha})^i \bar{c} \bar{\alpha} \bar{b} \bar{\alpha}$$

Proposition

$\mathcal{H}^*(z)$ is not context-free.

Example

$$ab\alpha c(ab)^i \alpha d ab\alpha c(ab)^i \alpha d \underbrace{ab\alpha c\alpha \bar{a}\bar{d}\bar{\alpha}\bar{d}\bar{\alpha}}_{=z} (\bar{b}\bar{\alpha})^i \bar{c}\bar{\alpha}\bar{b}\bar{\alpha}$$

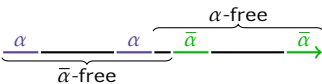
Proposition

$\mathcal{H}^*(z)$ is not context-free.

Questions

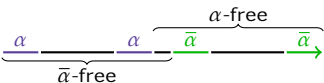
- i.) Is $\mathcal{H}^*(w) \stackrel{?}{\in} \text{REG}$ decidable?
- ii.) Does $w \in \Sigma^*$ exist such that $\mathcal{H}^*(w)$ is context-free but not regular?

Non-crossing Words

w is non-crossing \iff 

e. g., $z = ab\alpha c\alpha\bar{\alpha}\bar{d}\bar{\alpha}\bar{d}\bar{\alpha}$ is non-crossing

Non-crossing Words

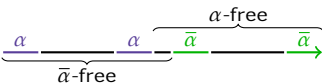
w is non-crossing \iff  The diagram shows a horizontal line with four segments. The first two segments are labeled with the Greek letter alpha (α) in purple. A bracket underneath the first two segments is labeled alpha-free with a bar (ā-free). The last two segments are labeled with alpha with a bar (ā) in green. A bracket above the last two segments is labeled alpha-free.

e. g., $z = abac\alpha\bar{a}\bar{d}\bar{a}\bar{d}\bar{a}$ is non-crossing

Proposition

Every word in $\mathcal{H}^*(w)$ is non-crossing if w is non-crossing.

Non-crossing Words

w is non-crossing \iff 

e. g., $z = abac\alpha\bar{a}\bar{d}\bar{a}\bar{d}\bar{a}$ is non-crossing

Proposition

Every word in $\mathcal{H}^*(w)$ is non-crossing if w is non-crossing.

Theorem (main result)

For non-crossing $w \in \Sigma^*$

- i.) $\mathcal{H}^*(w) \stackrel{?}{\in} \text{REG}$ is decidable.
- ii.) $\mathcal{H}^*(w)$ is either regular or not context-free.

Non-regularity of $\mathcal{H}^*(z)$

Let w be non-crossing.

$P_\alpha(w) = \{p_0, \dots, p_m\}$ s. t. $p_i\alpha$ is a prefix of w .

Non-regularity of $\mathcal{H}^*(z)$

Let w be non-crossing.

$P_\alpha(w) = \{p_0, \dots, p_m\}$ s. t. $p_i\alpha$ is a prefix of w .

$S_{\bar{\alpha}}(w) = \{\bar{s}_0, \dots, \bar{s}_n\}$ s. t. $\bar{\alpha}\bar{s}_i$ is a suffix of w .

Non-regularity of $\mathcal{H}^*(z)$

Let w be non-crossing.

$P_\alpha(w) = \{p_0, \dots, p_m\}$ s. t. $p_i\alpha$ is a prefix of w .

$S_{\bar{\alpha}}(w) = \{\bar{s}_0, \dots, \bar{s}_n\}$ s. t. $\bar{\alpha}\bar{s}_i$ is a suffix of w .

Example

$z = \alpha b \alpha c \alpha \bar{\alpha} \bar{d} \bar{\alpha} \bar{d} \bar{\alpha}$:

$$P_\alpha(z) = \{\varepsilon, \alpha b, \alpha b \alpha c\} \quad S_{\bar{\alpha}}(z) = \{\varepsilon, \bar{d} \bar{\alpha}, \bar{d} \bar{\alpha} \bar{d} \bar{\alpha}\}$$

Non-regularity of $\mathcal{H}^*(z)$

Let w be non-crossing.

$P_\alpha(w) = \{p_0, \dots, p_m\}$ s. t. $p_i\alpha$ is a prefix of w .

$S_{\bar{\alpha}}(w) = \{\bar{s}_0, \dots, \bar{s}_n\}$ s. t. $\bar{\alpha}\bar{s}_i$ is a suffix of w .

Example

$z = \alpha b \alpha c \alpha \bar{\alpha} \bar{d} \bar{\alpha} \bar{d} \bar{\alpha}$:

$P_\alpha(z) = \{\varepsilon, \alpha b, \alpha b \alpha c\}$ $S_{\bar{\alpha}}(z) = \{\varepsilon, \bar{d} \bar{\alpha}, \bar{d} \bar{\alpha} \bar{d} \bar{\alpha}\}$

Lemma

If $m, n \geq 1$, then $\mathcal{H}^*(w) = \{w\} \cup \bigcup_{1 \leq i \leq m} \mathcal{H}^*(w\bar{p}_i) \cup \bigcup_{1 \leq i \leq n} \mathcal{H}^*(s_i w)$.

Non-regularity of $\mathcal{H}^*(z)$

Let w be non-crossing.

$P_\alpha(w) = \{p_0, \dots, p_m\}$ s. t. $p_i\alpha$ is a prefix of w .

$S_{\bar{\alpha}}(w) = \{\bar{s}_0, \dots, \bar{s}_n\}$ s. t. $\bar{\alpha}\bar{s}_i$ is a suffix of w .

Example

$z = \alpha b \alpha c \alpha \bar{\alpha} \bar{d} \bar{\alpha} \bar{d} \bar{\alpha}$:

$$P_\alpha(z) = \{\varepsilon, \alpha b, \alpha b \alpha c\} \quad S_{\bar{\alpha}}(z) = \{\varepsilon, \bar{d} \bar{\alpha}, \bar{d} \bar{\alpha} \bar{d} \bar{\alpha}\}$$

Lemma

If $m, n \geq 1$, then $\mathcal{H}^*(w) = \{w\} \cup \bigcup_{1 \leq i \leq m} \mathcal{H}^*(w\bar{p}_i) \cup \bigcup_{1 \leq i \leq n} \mathcal{H}^*(s_i w)$.

Example

$$\mathcal{H}^*(z) = \{z\} \cup \mathcal{H}^*(z\bar{b}\bar{\alpha}) \cup \mathcal{H}^*(z\bar{c}\bar{\alpha}\bar{b}\bar{\alpha}) \cup \mathcal{H}^*(\alpha dz) \cup \mathcal{H}^*(\alpha d \alpha dz)$$

Non-regularity of $\mathcal{H}^*(z)$

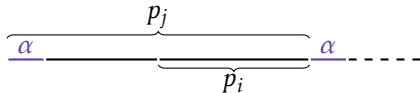
Lemma

If p_i is a suffix of p_j , then $\mathcal{H}^*(w\bar{p}_j) \subseteq \mathcal{H}^*(w\bar{p}_i)$.

Non-regularity of $\mathcal{H}^*(z)$

Lemma

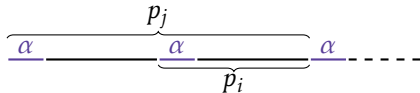
If p_i is a suffix of p_j , then $\mathcal{H}^*(w\bar{p}_j) \subseteq \mathcal{H}^*(w\bar{p}_i)$.



Non-regularity of $\mathcal{H}^*(z)$

Lemma

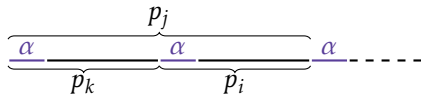
If p_i is a suffix of p_j , then $\mathcal{H}^*(w\bar{p}_j) \subseteq \mathcal{H}^*(w\bar{p}_i)$.



Non-regularity of $\mathcal{H}^*(z)$

Lemma

If p_i is a suffix of p_j , then $\mathcal{H}^*(w\bar{p}_j) \subseteq \mathcal{H}^*(w\bar{p}_i)$.

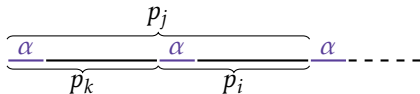


$$\Rightarrow p_j = p_k p_i$$

Non-regularity of $\mathcal{H}^*(z)$

Lemma

If p_i is a suffix of p_j , then $\mathcal{H}^*(w\bar{p}_j) \subseteq \mathcal{H}^*(w\bar{p}_i)$.



$$\Rightarrow p_j = p_k p_i$$

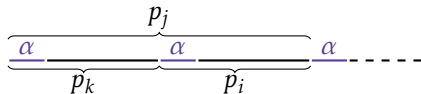
$$\Rightarrow w\bar{p}_i \rightarrow_{\mathcal{R}} w\bar{p}_i \bar{p}_k = w\bar{p}_j$$

$$\Rightarrow w\bar{p}_j \in \mathcal{H}^*(w\bar{p}_i)$$

Non-regularity of $\mathcal{H}^*(z)$

Lemma

If p_i is a suffix of p_j , then $\mathcal{H}^*(w\bar{p}_j) \subseteq \mathcal{H}^*(w\bar{p}_i)$.



$$\Rightarrow p_j = p_k p_i$$

$$\Rightarrow w\bar{p}_i \rightarrow_{\mathcal{R}} w\bar{p}_i \bar{p}_k = w\bar{p}_j$$

$$\Rightarrow w\bar{p}_j \in \mathcal{H}^*(w\bar{p}_i)$$

Lemma

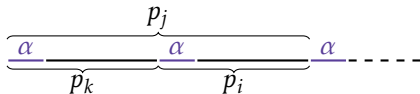
Let $P = \{p_1, \dots, p_m\} \setminus \{p_1, \dots, p_m\}^2$, $S = \{s_1, \dots, s_m\} \setminus \{s_1, \dots, s_m\}^2$, and $p_1 \neq s_1$.

- ▶ $\mathcal{H}^*(w) = \{w\} \cup \bigcup_{p \in P} \mathcal{H}^*(w\bar{p}) \cup \bigcup_{s \in S} \mathcal{H}^*(sw)$ and the union is disjoint.

Non-regularity of $\mathcal{H}^*(z)$

Lemma

If p_i is a suffix of p_j , then $\mathcal{H}^*(w\bar{p}_j) \subseteq \mathcal{H}^*(w\bar{p}_i)$.



$$\Rightarrow p_j = p_k p_i$$

$$\Rightarrow w\bar{p}_i \rightarrow_{\mathcal{R}} w\bar{p}_i \bar{p}_k = w\bar{p}_j$$

$$\Rightarrow w\bar{p}_j \in \mathcal{H}^*(w\bar{p}_i)$$

Lemma

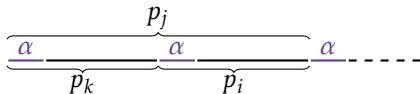
Let $P = \{p_1, \dots, p_m\} \setminus \{p_1, \dots, p_m\}^2$, $S = \{s_1, \dots, s_m\} \setminus \{s_1, \dots, s_m\}^2$, and $p_1 \neq s_1$.

- ▶ $\mathcal{H}^*(w) = \{w\} \cup \bigcup_{p \in P} \mathcal{H}^*(w\bar{p}) \cup \bigcup_{s \in S} \mathcal{H}^*(sw)$ and the union is disjoint.
- ▶ $\mathcal{H}^*(w)$ is regular iff $\forall p \in P: \mathcal{H}^*(w\bar{p})$ is regular and $\forall \bar{s} \in S: \mathcal{H}^*(s\bar{s})$ is regular.

Non-regularity of $\mathcal{H}^*(z)$

Lemma

If p_i is a suffix of p_j , then $\mathcal{H}^*(w\bar{p}_j) \subseteq \mathcal{H}^*(w\bar{p}_i)$.



$$\Rightarrow p_j = p_k p_i$$

$$\Rightarrow w\bar{p}_i \rightarrow_{\mathcal{R}} w\bar{p}_i \bar{p}_k = w\bar{p}_j$$

$$\Rightarrow w\bar{p}_j \in \mathcal{H}^*(w\bar{p}_i)$$

Lemma

Let $P = \{p_1, \dots, p_m\} \setminus \{p_1, \dots, p_m\}^2$, $S = \{s_1, \dots, s_m\} \setminus \{s_1, \dots, s_m\}^2$, and $p_1 \neq s_1$.

- ▶ $\mathcal{H}^*(w) = \{w\} \cup \bigcup_{p \in P} \mathcal{H}^*(w\bar{p}) \cup \bigcup_{s \in S} \mathcal{H}^*(sw)$ and the union is disjoint.
- ▶ $\mathcal{H}^*(w)$ is regular iff $\forall p \in P: \mathcal{H}^*(w\bar{p})$ is regular and $\forall s \in S: \mathcal{H}^*(sw)$ is regular.

Example

$$z = ab\alpha c\alpha \bar{a}\bar{d}\bar{a}\bar{d}\bar{a}:$$

$$\mathcal{H}^*(z) = \{z\} \cup \mathcal{H}^*(z\bar{b}\bar{a}) \cup \mathcal{H}^*(z\bar{c}\bar{a}\bar{b}\bar{a}) \cup \mathcal{H}^*(\alpha dz) \cup \mathcal{H}^*(\alpha d\alpha dz)$$

Non-regularity of $\mathcal{H}^*(z)$

Theorem

Assume $p_1 = s_1$. $\mathcal{H}^*(w)$ is regular iff

1. $p_i \in \overline{S_{\bar{\alpha}}(w)^*}$ or $\overline{S_{\bar{\alpha}}(w)} \subseteq \{p_1, \dots, p_i\}^*$ for all $i \leq m$ and
2. $s_i \in P_{\alpha}(w)^*$ or $P_{\alpha}(w) \subseteq \{s_1, \dots, s_i\}^*$ for all $i \leq n$.

Non-regularity of $\mathcal{H}^*(z)$

Theorem

Assume $p_1 = s_1$. $\mathcal{H}^*(w)$ is regular iff

1. $p_i \in \overline{S_{\bar{\alpha}}(w)}^*$ or $\overline{S_{\bar{\alpha}}(w)} \subseteq \{p_1, \dots, p_i\}^*$ for all $i \leq m$ and
2. $s_i \in P_{\alpha}(w)^*$ or $P_{\alpha}(w) \subseteq \{s_1, \dots, s_i\}^*$ for all $i \leq n$.

Example

$z = \alpha b \alpha c \alpha \bar{\alpha} \bar{d} \bar{\alpha} \bar{d} \bar{\alpha}$:

$\mathcal{H}^*(z)$ is regular iff $\mathcal{H}^*(z\bar{b}\bar{\alpha})$, $\mathcal{H}^*(z\bar{c}\bar{\alpha}\bar{b}\bar{\alpha})$, and $\mathcal{H}^*(\alpha dz)$ are regular.

1. $P_{\alpha}(\alpha dz) = \{\varepsilon, \alpha d, \alpha d \alpha b, \alpha d \alpha b \alpha c\}$ $\overline{S_{\bar{\alpha}}(\alpha dz)} = \{\varepsilon, \alpha d, \alpha d \alpha d\}$

Non-regularity of $\mathcal{H}^*(z)$

Theorem

Assume $p_1 = s_1$. $\mathcal{H}^*(w)$ is regular iff

1. $p_i \in \overline{S_{\bar{\alpha}}(w)}^*$ or $\overline{S_{\bar{\alpha}}(w)} \subseteq \{p_1, \dots, p_i\}^*$ for all $i \leq m$ and
2. $s_i \in P_{\alpha}(w)^*$ or $P_{\alpha}(w) \subseteq \{s_1, \dots, s_i\}^*$ for all $i \leq n$.

Example

$z = \alpha b \alpha c \alpha \bar{\alpha} \bar{d} \bar{\alpha} \bar{d} \bar{\alpha}$:

$\mathcal{H}^*(z)$ is regular iff $\mathcal{H}^*(z\bar{b}\bar{\alpha})$, $\mathcal{H}^*(z\bar{c}\bar{\alpha}\bar{b}\bar{\alpha})$, and $\mathcal{H}^*(\alpha dz)$ are regular.

1. $P_{\alpha}(\alpha dz) = \{\varepsilon, \alpha d, \alpha d \alpha b, \alpha d \alpha b \alpha c\}$ $\overline{S_{\bar{\alpha}}(\alpha dz)} = \{\varepsilon, \alpha d, \alpha d \alpha d\}$
 $\Rightarrow \mathcal{H}^*(\alpha dz)$ is regular. $\mathcal{H}^*(\alpha dz)$ is regular.

Non-regularity of $\mathcal{H}^*(z)$

Theorem

Assume $p_1 = s_1$. $\mathcal{H}^*(w)$ is regular iff

1. $p_i \in \overline{S_{\bar{\alpha}}(w)^*}$ or $\overline{S_{\bar{\alpha}}(w)} \subseteq \{p_1, \dots, p_i\}^*$ for all $i \leq m$ and
2. $s_i \in P_{\alpha}(w)^*$ or $P_{\alpha}(w) \subseteq \{s_1, \dots, s_i\}^*$ for all $i \leq n$.

Example

$z = ab\alpha c\alpha\bar{\alpha}\bar{d}\bar{\alpha}\bar{d}\bar{\alpha}$:

$\mathcal{H}^*(z)$ is regular iff $\mathcal{H}^*(z\bar{b}\bar{\alpha})$, $\mathcal{H}^*(z\bar{c}\bar{\alpha}\bar{b}\bar{\alpha})$, and $\mathcal{H}^*(\alpha dz)$ are regular.

1. $P_{\alpha}(\alpha dz) = \{\varepsilon, \alpha d, \alpha d\alpha b, \alpha d\alpha b\alpha c\}$ $\overline{S_{\bar{\alpha}}(\alpha dz)} = \{\varepsilon, \alpha d, \alpha d\alpha d\}$
 $\Rightarrow \mathcal{H}^*(\alpha dz)$ is regular. $\mathcal{H}^*(\alpha dz)$ is regular.
2. $P_{\alpha}(z\bar{b}\bar{\alpha}) = \{\varepsilon, \alpha b, \alpha b\alpha c\}$ $\overline{S_{\bar{\alpha}}(z\bar{b}\bar{\alpha})} = \{\varepsilon, \alpha b, \alpha b\alpha d, \alpha b\alpha d\alpha d\}$

Non-regularity of $\mathcal{H}^*(z)$

Theorem

Assume $p_1 = s_1$. $\mathcal{H}^*(w)$ is regular iff

1. $p_i \in \overline{S_{\bar{\alpha}}(w)^*}$ or $\overline{S_{\bar{\alpha}}(w)} \subseteq \{p_1, \dots, p_i\}^*$ for all $i \leq m$ and
2. $s_i \in P_{\alpha}(w)^*$ or $P_{\alpha}(w) \subseteq \{s_1, \dots, s_i\}^*$ for all $i \leq n$.

Example

$z = ab\alpha c\alpha\bar{\alpha}\bar{d}\bar{\alpha}\bar{d}\bar{\alpha}$:

$\mathcal{H}^*(z)$ is regular iff $\mathcal{H}^*(z\bar{b}\bar{\alpha})$, $\mathcal{H}^*(z\bar{c}\bar{\alpha}\bar{b}\bar{\alpha})$, and $\mathcal{H}^*(\alpha dz)$ are regular.

1. $P_{\alpha}(\alpha dz) = \{\varepsilon, \alpha d, \alpha d\alpha b, \alpha d\alpha b\alpha c\}$ $\overline{S_{\bar{\alpha}}(\alpha dz)} = \{\varepsilon, \alpha d, \alpha d\alpha d\}$
 $\Rightarrow \mathcal{H}^*(\alpha dz)$ is regular. $\mathcal{H}^*(\alpha dz)$ is regular.
2. $P_{\alpha}(z\bar{b}\bar{\alpha}) = \{\varepsilon, \alpha b, \alpha b\alpha c\}$ $\overline{S_{\bar{\alpha}}(z\bar{b}\bar{\alpha})} = \{\varepsilon, \alpha b, \alpha b\alpha d, \alpha b\alpha d\alpha d\}$
 $\Rightarrow \mathcal{H}^*(z\bar{b}\bar{\alpha})$ is not regular.

Non-regularity of $\mathcal{H}^*(z)$

Theorem

Assume $p_1 = s_1$. $\mathcal{H}^*(w)$ is regular iff

1. $p_i \in \overline{S_{\bar{\alpha}}(w)^*}$ or $\overline{S_{\bar{\alpha}}(w)} \subseteq \{p_1, \dots, p_i\}^*$ for all $i \leq m$ and
2. $s_i \in P_{\alpha}(w)^*$ or $P_{\alpha}(w) \subseteq \{s_1, \dots, s_i\}^*$ for all $i \leq n$.

Example

$z = ab\alpha c\alpha\bar{\alpha}\bar{d}\bar{\alpha}\bar{d}\bar{\alpha}$:

$\mathcal{H}^*(z)$ is regular iff $\mathcal{H}^*(z\bar{b}\bar{\alpha})$, $\mathcal{H}^*(z\bar{c}\bar{\alpha}\bar{b}\bar{\alpha})$, and $\mathcal{H}^*(\alpha dz)$ are regular.

1. $P_{\alpha}(\alpha dz) = \{\varepsilon, \alpha d, \alpha d\alpha b, \alpha d\alpha b\alpha c\}$ $\overline{S_{\bar{\alpha}}(\alpha dz)} = \{\varepsilon, \alpha d, \alpha d\alpha d\}$
 $\Rightarrow \mathcal{H}^*(\alpha dz)$ is regular. $\mathcal{H}^*(\alpha dz)$ is regular.
2. $P_{\alpha}(z\bar{b}\bar{\alpha}) = \{\varepsilon, \alpha b, \alpha b\alpha c\}$ $\overline{S_{\bar{\alpha}}(z\bar{b}\bar{\alpha})} = \{\varepsilon, \alpha b, \alpha b\alpha d, \alpha b\alpha d\alpha d\}$
 $\Rightarrow \mathcal{H}^*(z\bar{\alpha}\bar{b})$ is not regular.
 $\Rightarrow \mathcal{H}^*(z)$ is not regular.

Questions?

Thank you!