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Coalitions in Hedonic Games

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Contents

Central concepts

- Finite set N of players
- Coalition = non-empty subset of N
- Partition Π divides N into disjoint coalitions
- $\Pi(i)$ denotes coalition in Π containing player $i \in N$

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- A coalition S blocks a partition Π, if all players $i \in S$ have $\Pi(i) \prec_i S$ and hence strictly prefer being in S to being in current coalition $\Pi(i)$.

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Central definition:
A partition Π is core stable, if there is no blocking coalition S.

Closely related:
weakly blocking coalition (no player worse off; at least one player better off)
strongly core stable partition (no weakly blocking coalition)

Example

Three players a, b, c

Preferences of player a: $ab > ac > a > abc$
Preferences of player b: $bc > ab > b > abc$
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Fact.
This simple cyclic 3-player game does not allow a core stable partition.

Central algorithmic questions (1)

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Given: set N with all the preferences of the players.
Decide whether there exists some core stable partition.

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Does there exist a partition Π of the players,
such that for all coalitions S:
S does not block Π?

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Decide whether there exists some core stable partition.

Or equivalently:
Does there exist a partition Π of the players,
such that for all coalitions S:
S does not block Π?

Or even shorter: $\exists\Pi\,\forall S : \neg\,(S \text{ blocks } \Pi)$

Note: not a simple existential NP-formulation $\exists x : P(x)$
but a Σ_2^p formulation with two quantifiers $\exists x \forall y : P(x,y)$

Central algorithmic questions (2)

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Depending on the exact way of representing the instance, this question will typically be

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- or complete for class NP
- or complete for class Σ_2^p
- or somewhere inbetween
- or perhaps computationally trivial (and always have answer YES)

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Observation:
If the companion problem is solvable in polynomial time,
then the main problem is contained in NP.

The Ballester encoding (1)

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If $\{i\} \succ_i S$, then coalition S can never show up in a core stable partition.

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For every player i, coalitions S with $S \succ_i \{i\}$ are explicitly listed and ranked.
The remaining coalitions are unacceptable and ignored.

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Observation:
Under Ballester encoding, companion problem $\exists S$: (S blocks Π) is in P.
Under Ballester encoding, $\exists \Pi \, \forall S : \neg$ (S blocks Π) lies in NP.

The Ballester encoding (2)

[Karp, 1971] showed EXACT COVER BY 3-SETS (XC3) to be NP-complete:

Given: ground set X; system $\mathcal{S}$ of 3-element subsets of X.
Question: Does there exist a partition of X that only uses parts from $\mathcal{S}$?

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Theorem.
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Proof.
Every element $x \in X$ in XC3 instance becomes three players x, x', x''.

Player x: (all sets $S \in \mathcal{S}$ with $x \in S$ equally) $> \; xx'' \; > \; xx' \; > \; x$
Player x': $xx' > x'x'' > x'$
Player x'': $x'x'' > xx'' > x''$ $\square$

Preferences from graphs (1)

[Dimitrov, Borm, Hendrickx, Sung, 2006]
propose preference structures based on directed graphs $G = (N, A)$.

An arc (x, y) from player x to y means that x considers y a friend.
Set F_x contains the friends of player x, and set E_x contains his enemies

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Definition.
Player x prefers S to T ("$S \succeq_x T$"), if and only if

(i) $|S \cap F_x| > |T \cap F_x|$, or

(ii) $|S \cap F_x| = |T \cap F_x|$ and $|S \cap E_x| \leq |T \cap E_x|$

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There exists a j such that players in $C_j \cap S$ have no friends in $S - C_j$.

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Hence they cannot have additional enemies. Hence $C_j = S$. $\square$

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Open problem.
Complexity of companion problem $\exists S$: (S blocks Π) for this friend-oriented scenario.

More preferences from graphs (1)

[Dimitrov & al 2006] also discuss graph-based enemy-oriented preferences.

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Observation.
If player x does not like player y, then in any core stable partition x and y must not be in the same coalition.

Consequence.
Assume that friendship is mutual/symmetric, and use undirected graphs.
In core stable partition, every coalition is a clique.

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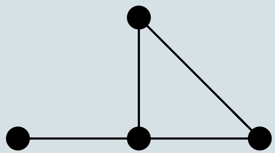
Theorem.
Under enemy-oriented preferences, companion-problem $\exists S$: (S blocks Π) is NP-complete.
Proof.
Decide existence of k-clique in graph G: for every vertex v create $k-2$ new vertices that together with v form a $(k-1)$-clique.
Consider partition that consists of all these $(k-1)$-cliques. □

More preferences from graphs (3)

Open problem.
Complexity of deciding existence of strongly core stable partitions under enemy-oriented preferences.

- Known: NP-hard; contained in Σ_2^p



Anonymous preferences (1)

Every player is indifferent among coalitions of the same size.

- Example: for a chess-player all coalitions of even size might be fine, and all coalitions of odd size might be unacceptable.

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- Companion-problem $\exists S$: (S blocks Π) is polynomially solvable.
(Are there ≥ 16 players who would prefer to be in a coalition of size 16?)

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- Companion-problem $\exists S$: (S blocks Π) is polynomially solvable.
(Are there ≥ 16 players who would prefer to be in a coalition of size 16?)

Theorem. [Ballester, 2004]
Under anonymous preferences, deciding the existence of a core stable partition is NP-complete.

Anonymous preferences (2)

Closely related scenario: every player has black-and-white view of the world; (equally) likes some coalition sizes, and (equally) hates remaining ones.

- A partition is wonderfully stable, if each player likes size of his coalition

Theorem. [Darmann, Elkind, Kurz, Lang, Schauer, Woeginger, 2012]
Under anonymous black-and-white preferences, deciding existence of a wonderfully stable partition is NP-complete even if every player likes only two sizes and hates all the remaining ones.

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Open problem.
Complexity of deciding existence of core stable partitions under anonymous preferences, if every player x

- (equally) likes the sizes s in the interval $\ell(x) \leq s \leq r(x)$ and
- (equally) hates all sizes outside this interval.

Gale-Shapley matching (1)

- Half the players are female; half the players are male
- The only acceptable coalitions are man-woman pairs
- Every player ranks all players of other sex (no ties)
- Slang: stable matching $\equiv$ core stable partition

David Gale and Lloyd Stowell Shapley:
"College Admissions and the Stability of Marriage"
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Theorem.
A stable matching always exists, and can be found in polynomial time.

Gale-Shapley matching (2)

Note: The most trivial and most straightforward approach does not work.

Iterative improvement:
Repeatedly find a blocking pair w, m that is currently matched as (w, m') and (w', m). Improve the matching by using (w, m) and (w', m') instead.

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Example:

A:	XZWY	AW, BX, CZ, DY	A and Z are blocking
B:	YWXZ	AZ, BX, CW, DY	C and Y are blocking
C:	ZXYW	AZ, BX, CY, DW	B and W are blocking
D:	WYZX	AZ, BW, CY, DX	D and Z are blocking
		AX, BW, CY, DZ	C and X are blocking
W:	ACBD	AY, BW, CX, DZ	A and W are blocking
X:	BDCA	AW, BY, CX, DZ	D and Y are blocking
Y:	CADB	AW, BZ, CX, DY	B and X are blocking
Z:	DBAC	AW, BX, CZ, DY	etc. etc. etc.

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Some closely related scenarios (coalitions of size 2 or 1):

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  Stable matching may not exist; polynomially solvable [Irving, 1985]

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- Stable roommates: uni-sex version;
  Stable matching may not exist; polynomially solvable [Irving, 1985]
- Stable roommates with ties in preference lists:
  Deciding existence of stable matching is NP-complete [Ronn, 1990]

Gale-Shapley variants (2)

Generalizations of Gale-Shapley to three dimensions (man, woman, dog) are usually messy

- There are instances without stable matching [Alkan, 1988]
- Deciding existence of stable matching is NP-complete [Ng, Hirschberg, 1991]

Geometric preferences (1)

Every player is a point in a metric space, or in a Eudlidean space.
Player x ranks coalition S according to $\sum_{s \in S} d(x,s)$ (small=good)

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The metric space version of the stable roommate problem always has a core stable partition.

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The metric space version of the stable roommate problem always has a core stable partition.
Proof.
Pick closest pair as coalition. Repeat on remaining points. $\square$

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Proof.
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Fact. [Arkin, Bae, Efrat, Okamoto, Mitchell, Polishchuk, 2009]
There exist instances of the metric space version of the 3-dimensional roommate problem (only coalitions of size 3 acceptable) that do not have a core stable partition.

Geometric preferences (2)

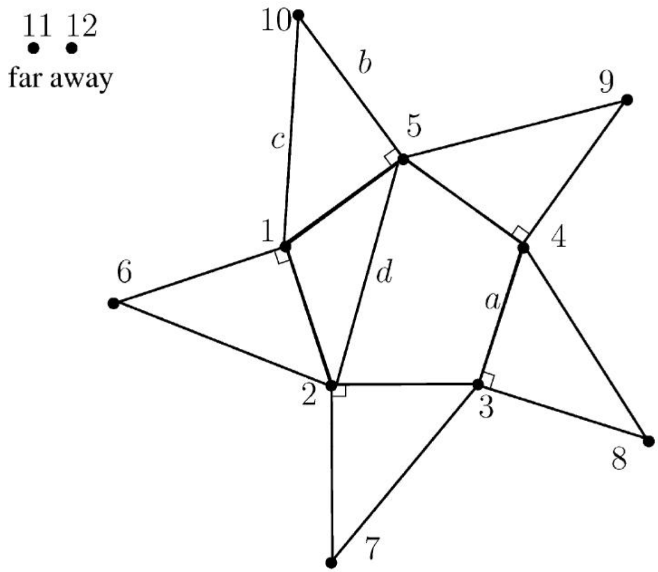

Fig. 1. (1, 2, 3, 4, 5) is a regular pentagon. (1, 5, 10), (1, 2, 6), (2, 3, 7), (3, 4, 8), and (4, 5, 9) are congruent right triangles. The points 11 and 12 are far away from the others. $a < b < c < d$.

Geometric preferences (3)

Open problem.

Complexity of deciding existence of core stable partitions for this Euclidean version of the 3-dimensional roommate problem.

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Open problem.
Behavior of other Euclidean/metric variants where player x ranks coalitions S for instance according to maximum (or minimum) distance between point x and the points in coalition S.

Additive preferences (1)

Every player x assigns to every other player y a real value $v_x(y)$.
High positive value = very attracted; negative value = aversion

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Friend-oriented preferences from graphs:
if $(x, y) \in A$ then $v_x(y) = |N|$
if $(x, y) \notin A$ then $v_x(y) = -1$

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if $(x, y) \notin A$ then $v_x(y) = -1$
Enemy-oriented preferences from graphs:
if $(x, y) \in A$ then $v_x(y) = 1$
if $(x, y) \notin A$ then $v_x(y) = -|N|$

Additive preferences (2)

Theorem. [Sung, Dimitrov, 2007]
Under additive preferences, the companion-problem $\exists S$: (S blocks Π) is NP-complete. $\square$

- Hard even for special case of enemy-oriented preferences from graphs.
- Hard even for special case of symmetric additive preferences where $v_x(y) = v_y(x)$.

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- Hard even for special case of symmetric additive preferences where $v_x(y) = v_y(x)$.

Theorem. [Woeginger, 2012]
Under additive preferences, $\exists \Pi\ \forall S : \neg$ (S blocks Π) is Σ_2^p-complete.

Sketch of proof. On the following slides.

Additive preferences (3)

Problem 2-QUANTIFIED 3-DNF-SAT captures the full difficulty of class Σ_2^p:

Given: Sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ of Boolean variables; a Boolean formula $\phi(X, Y)$ over $X \cup Y$ in disjunctive normal form where each of the conjunctive clauses consists of exactly three distinct literals.

Question: Is $\exists x_1, \ldots, x_n \forall y_1, \ldots, y_n : \phi(X, Y)$ true?

Example:
$\exists x_1, x_2, x_3 \;\; \forall y_1, y_2, y_3 : \;\; x_1\overline{y_2}y_3 \vee \overline{x_3}y_1y_2 \vee y_1y_2y_3 \vee x_3y_2y_3$

Additive preferences (4)

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- For every y_i, create two players $P(y_i)$ and $P(\overline{y_i})$ that hate each other
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- Verification: If there is a truth setting of Y variables that makes all clauses false, then the medium happy X players, half of the Y players (derived from truth setting), and all of the clause players form a blocking coalition.

Additive preferences (5)

	$P(x)$	$P(y)$	$P(c)$	Q_t	Q'_t	Q''_t	Q_f	R	R'
$P(x)$	$-\infty/0/\varepsilon$	0	0	1	0	$-\infty$	4	2	$-\infty$
$P(y)$	0	$-\infty/0$	0	$-\infty$	$-\infty$	$-\infty$	$-\infty$	2	$-\infty$
$P(c)$	$-2/0$	$-2/0$	0	6	$-\infty$	0	$-\infty$	5	$-\infty$
Q_t	1	$-\infty$	1	0	$m+n+1$	$n+2$	$-\infty$	$-\infty$	$-\infty$
Q'_t	ε	$-\infty$	$-\infty$	1	0	$-\infty$	$-\infty$	$-\infty$	$-\infty$
Q''_t	$-\infty$	$-\infty$	0	1	$-\infty$	0	$-\infty$	$-\infty$	$-\infty$
Q_f	1	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	0	$-\infty$	$-\infty$
R	2	2	2	$-\infty$	$-\infty$	$-\infty$	$-\infty$	0	$4n+2m-1$
R'	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	1	0

Additive preferences (6)

Open problem.
Complexity of deciding existence of strongly core stable partitions for additive preferences.

- Known: NP-hard [Sung, Dimitrov, 2010]; contained in class Σ_2^p

Final remarks

All in all:

- Rich and colorful area
- Algorithmic behavior surprisingly diverse

Not covered:

B-preferences and W-preferences
[Cechlarova & Hajdukova; many papers]