Improved Approximations for Ordered TSP on Near-Metric Graphs

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Overview

- Near-metric graphs
- Traveling salesman problem (TSP)
- Approximating the TSP in near-metric graphs
- Ordered TSP
- An old algorithm for Ordered TSP
- An improved algorithm

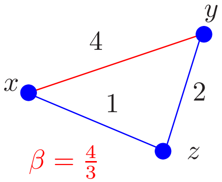
Near-metric graphs

Definition

An undirected (complete) graph $G = (V, E)$ with edge cost function *cost* satisfies the β-triangle inequality if

$$cost(x, y) \leq \beta \cdot (cost(x, z) + cost(z, y))$$

for all $x, y, z \in V$.



$\beta = 1$: Standard triangle inequality

$\beta > 1$: Relaxed triangle inequality

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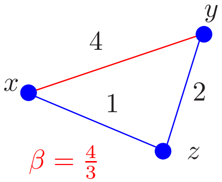

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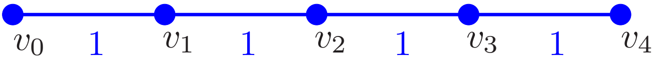
Relaxed triangle inequality – Properties

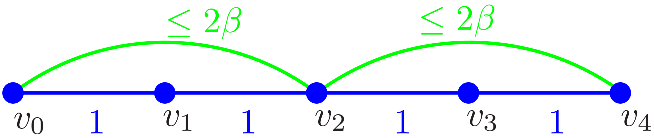


Lemma [Bandelt et al. 1994]

Let $v_0, v_1, \ldots, v_{2^k}$ be a path in G. Then

$$cost(v_0, v_{2^k}) \leq \beta^k \cdot \sum_{i=0}^{2^k-1} cost(v_i, v_{i+1})$$

Relaxed triangle inequality – Properties



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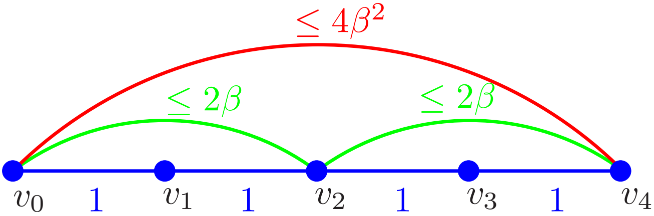

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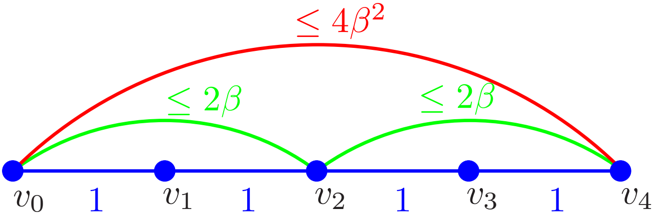

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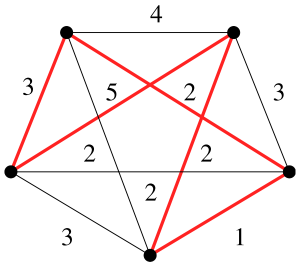
Definition of TSP

Traveling Salesman Problem (Δ_β-TSP)

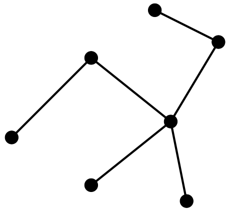
- Input: Undirected complete graph with edge costs
- Edge costs satisfy relaxed β-triangle inequality for $\beta > 1$
- Find a cheapest cycle visiting each vertex exactly once.



Strategy for approximating Δ-TSP ($\beta = 1$)

1. Find spanning subgraph with cost bounded by $\alpha \cdot \mathit{cost}(\mathit{Opt})$ for some α
2. Follow this subgraph to construct Hamiltonian cycle using jumps

⇒ Jumps do not increase cost due to triangle inequality



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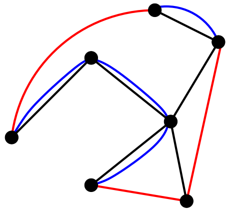

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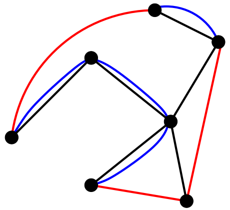

Strategy for approximating Δ_β-TSP ($\beta > 1$)

- Similar strategy, but:
- Jumps are expensive, their cost grows with the number of bypassed edges

$\Rightarrow$ Try to use short jumps

Example

Along a spanning tree, jumps of length 3 are sufficient

$\Rightarrow$ Approximation ratio $2\beta^2$

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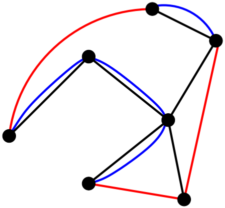

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Known approximation ratios for Δ_β-TSP

spanning subgraph	approx. ratio	best for
tree [Andreae & Bandelt 1995, Andreae 2001]	$\beta^2 + \beta$	$2 \leq \beta \leq 3$
biconnected graph [Bender & Chekuri 1999]	4β	$\beta \geq 3$
tree + path matching [Böckenhauer et al. 2000]	$\frac{3}{2}\beta^2$	$\beta \leq 2$

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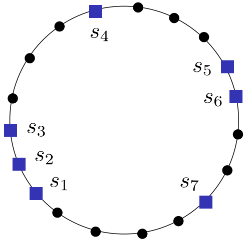
Definition of k-Δ_βOTSP

Ordered TSP (k-Δ_βOTSP)

Find a cheapest cycle visiting each vertex exactly once that in addition visits terminal vertices $s_1, \ldots, s_k$ in prescribed order.



Known approximation algorithm for k-Δ_βOTSP

Simple cycle algorithm
[Böckenhauer et al. 2006]

1. Build Hamiltonian cycle H
2. Color non-terminals periodically with $k + 1$ colors
3. Use non-terminals of color i to connect s_i with s_{i+1}
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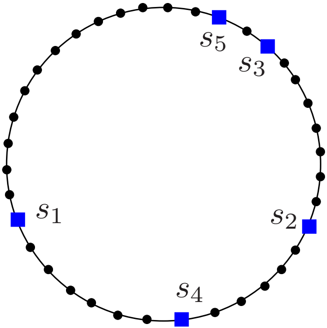

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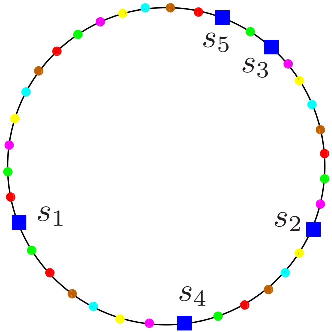

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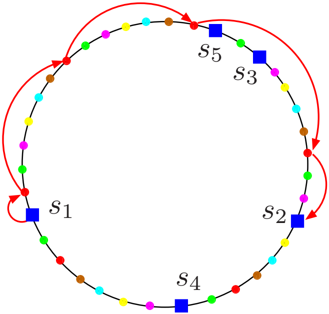

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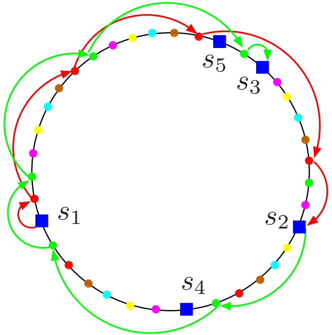

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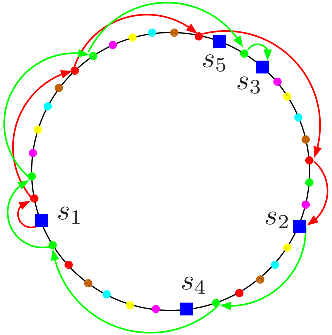

Approximation ratio of known k-Δ_βOTSP algorithms

Analysis of simple cycle algorithm

The simple cycle algorithm achieves approx. ratio of

$$\underbrace{(k+1)}_{\#\text{ rounds}} \cdot \underbrace{\beta^{1+\log_2(k-1)}}_{\text{log of jump length}} \cdot \underbrace{\min\left\{\frac{3}{2}\beta^2, \beta^2+\beta, 4\beta\right\}}_{\text{approx. of cycle}}$$

Improvement [Böckenhauer et al. 2010]

- Based on shortening Eulerian tour through doubled spanning tree
- Achieves approximation ratio of

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Improved cycle algorithm – Idea

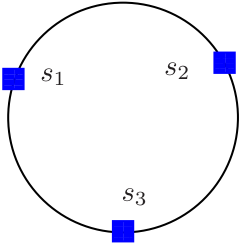
Idea

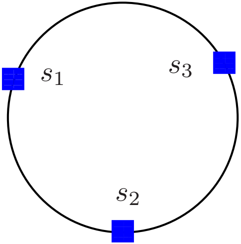
- Three terminals on the Hamiltonian cycle are always in correct order – either clockwise or counterclockwise

⇒ Connect triple of terminals in one round around the Hamiltonian cycle

⇒ Reduces number of rounds

⇒ Reduces jump lengths





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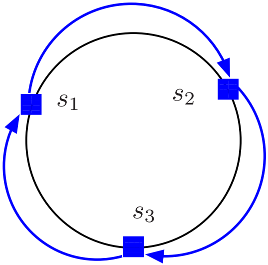

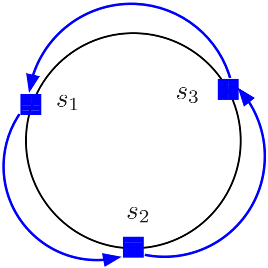

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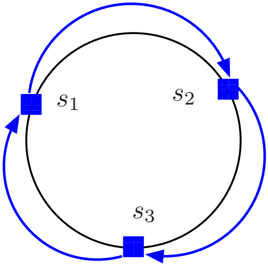

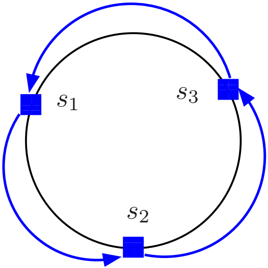

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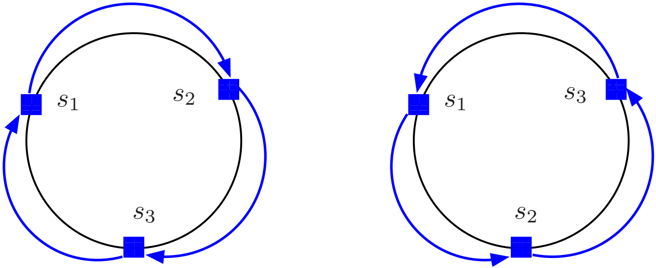

Improved cycle algorithm – Connecting triples of terminals

1. Build a Hamiltonian cycle H
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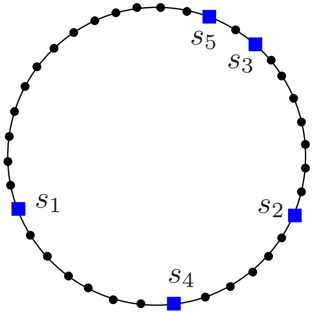

Observation

Each vertex of the graph appears in the cycle *at most* once. Some non-terminals are *not* connected.

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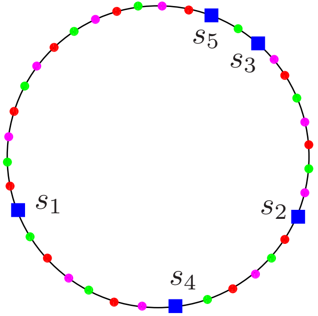

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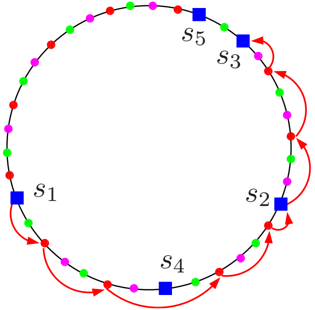

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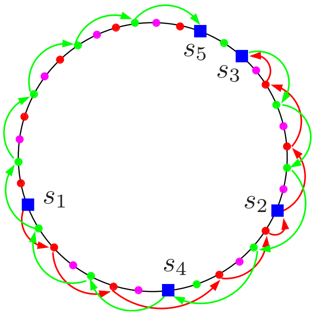

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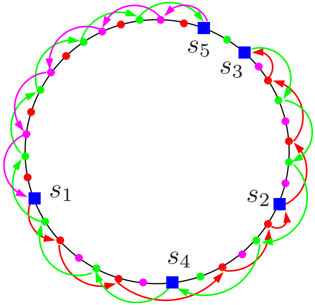

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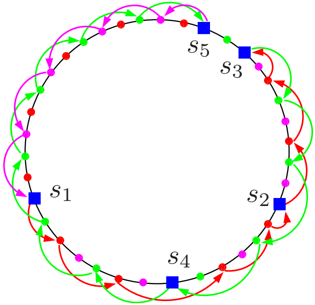

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Improved cycle algorithm – Including unused non-terminals

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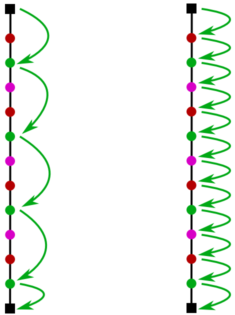
5. Extend two paths to cover remaining non-terminals



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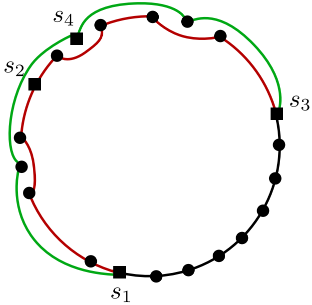

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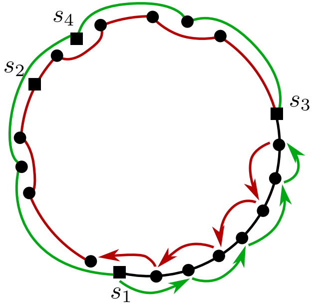
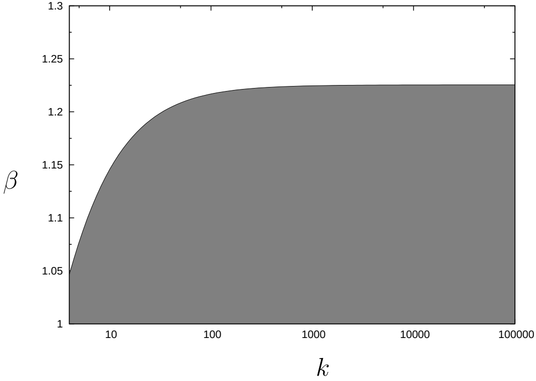


Approximation ratio of improved cycle algorithm

Theorem

The approximation ratio of the improved cycle algorithm is

$$\left\lceil \frac{k}{2} \right\rceil \cdot \beta^{\log_2(2\lceil k/2\rceil + k - 3)} \cdot \min\left\{\frac{3}{2}\beta^2, 4\beta, \beta^2 + \beta\right\}$$

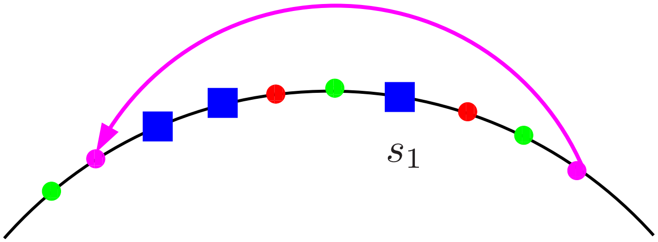


Further possible improvements

Observation

- Jump length $2\lceil k/2 \rceil + k - 3$ can only occur when bypassing s_1

$\Rightarrow$ Finding best starting position for the coloring might lower the jump length, especially for $k \ll n$



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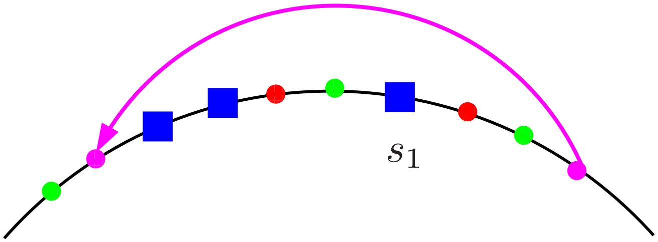

Thanks for your attention!