Collective Additive Tree Spanners of Bounded Tree-Breadth Graphs and Consequences

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Tree t-Spanner Problem

Given unweighted undirected graph $G = (V, E)$ and integers t, r.
Does G admit a spanning tree $T = (V, E')$ such that

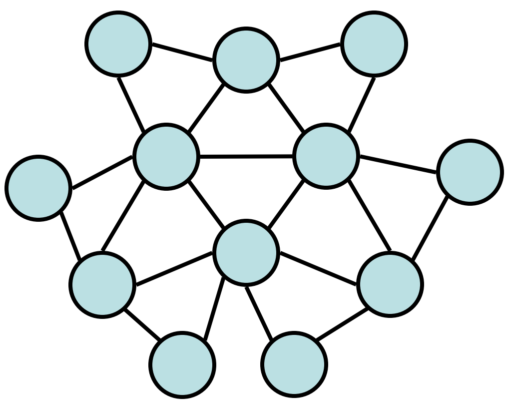
$$\forall\, u, v \in V, d_T(u,v) \leq t d_G(u,v)$$

(a multiplicative tree t-spanner of G)

or

$$\forall\, u, v \in V, d_T(u,v) \leq d_G(u,v) + r$$

(an additive tree r-spanner of G)?



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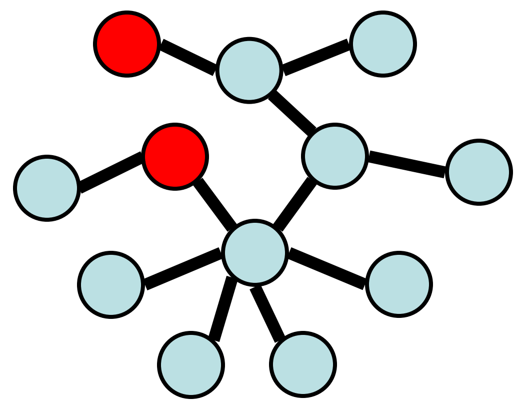


multiplicative tree 4- and additive tree 3-spanner of G

Sparse $\boldsymbol{t}$-Spanner Problem

Given unweighted undirected graph $G = (V, E)$ and integers t, m, r.
Does G admit a spanning graph $H = (V, E')$ with $|E'| \leq m$ such that

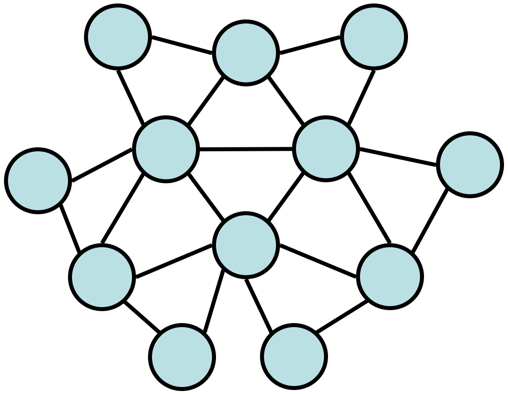
$$\forall\, u, v \in V, d_H(u, v) \leq t d_G(u, v)$$

(a multiplicative t-spanner of G)

or

$$\forall\, u, v \in V, d_H(u, v) \leq d_G(u, v) + r$$

(an additive r-spanner of G)?



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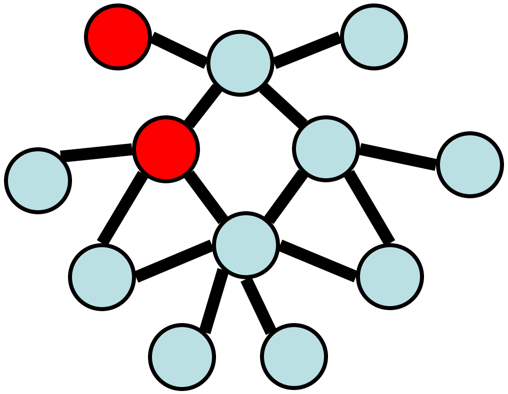


multiplicative 2- and additive 1-spanner of G

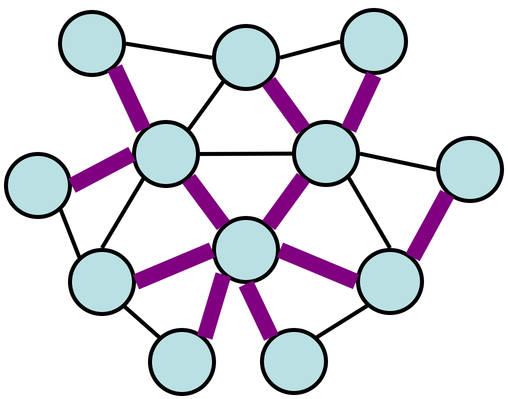
Collective Additive Tree r-Spanners Problem

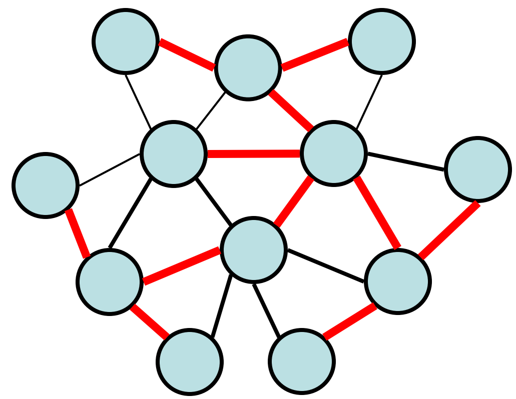
Given unweighted undirected graph $G=(V,E)$ and integers μ, r.

Does G admit a system of μ collective additive tree r*-spanners* $\{T_1, T_2, \dots, T_\mu\}$ such that

$$\forall\, u,v \in V, \exists\, 0 \le i \le \mu,\ d_{T_i}(u,v) \le d_G(u,v) + r$$

(a system of μ collective additive tree r-spanners of G)?





2 collective additive tree 2-spanners

Applications of Collective Tree Spanners

- Message routing in networks

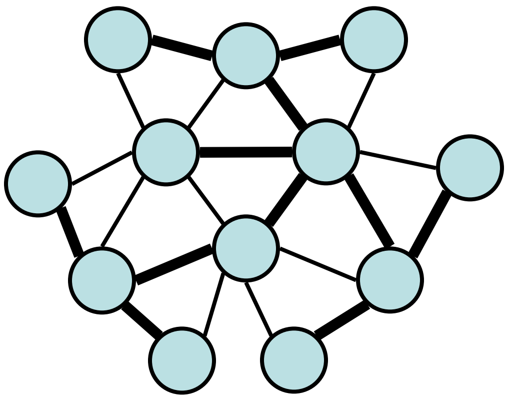
  Efficient routing scheme is known for trees but very hard for graphs. For any two nodes, we can route the message between them in one of the trees which approximates the distance between them.

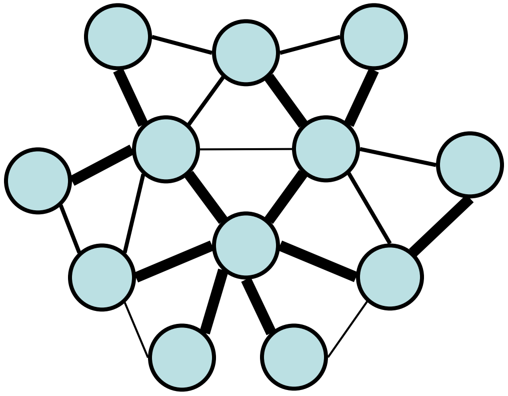
- Solution for sparse t-spanner problem

  If a graph admits a system of μ collective additive tree r-spanners, then the graph admits a sparse additive r-spanner with at most $\mu(n-1)$ edges, where n is the number of nodes.

- Approximate solution for problems on original graph

  Problems related to distances, e.g., approximate distance labeling schemes, etc…



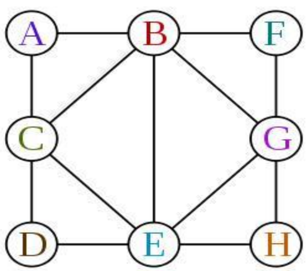


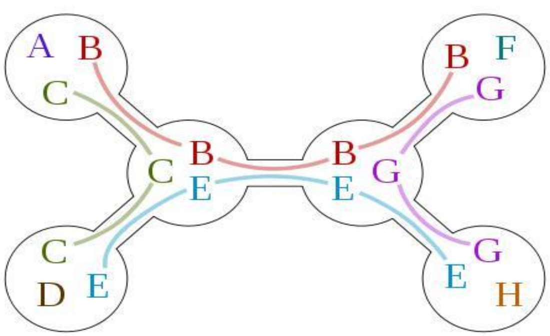
2 collective tree 2-spanners for G

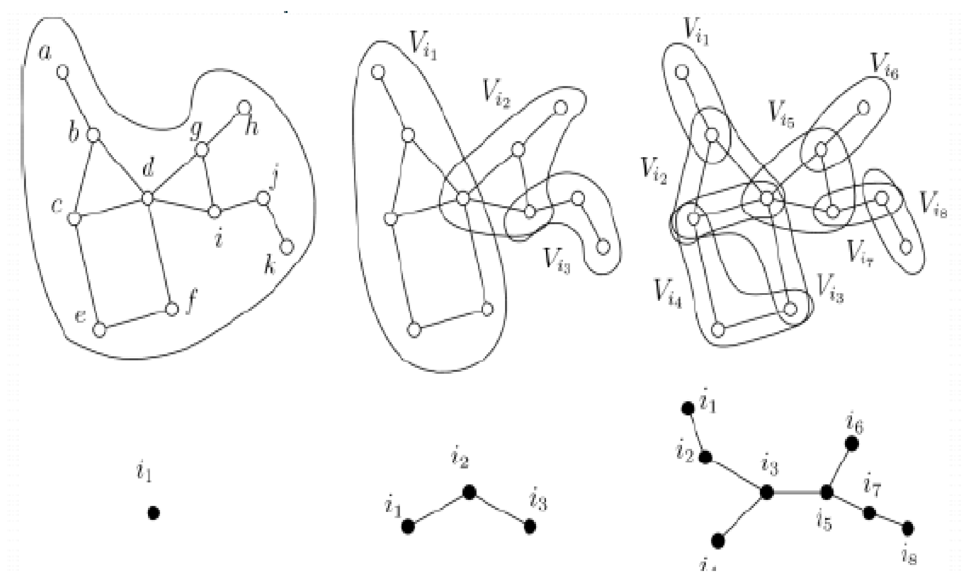
Tree-Decomposition

- Tree-decomposition $T(G)$ of a graph $G = (V, E)$ is a pair $(\{X_i : i \in I\}, T = (I, F))$ where $\{X_i : i \in I\}$ is a collection of subset of V (bags) and T is a tree whose nodes are the bags satisfying:

1) $\bigcup_{i \in I} X_i = V$
2) $\forall\, uv \in E, \exists\, i \in I\ s.t.\ u, v \in X_i$
3) $\forall\, v \in V, the\ set\ of\ bags\ \{i \in I, v \in X_i\}\ form\ a\ subtree\ T_v\ of\ T(G)$

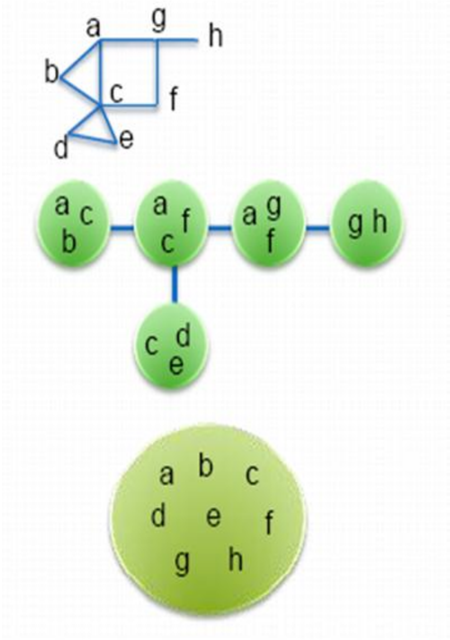






Tree-Decomposition and Graph Parameters

- Tree-width $tw(G)$:
  - Width of $T(G)$ is $\max_{i \in I} |X_i| - 1$
  - $tw(G)$: minimum width over all tree-decompositions
- Tree-length $tl(G)$:
  - Length of $T(G)$ is $\max_{i \in I} \max_{u,v \in X_i} d_G(u,v)$
  - $tl(G)$: minimum length over all tree-decompositions
- Tree-breadth $tb(G)$:
  - Breadth is minimum r such that $\forall i \in I, \exists v_i$ with $X_i \subseteq D_r(v_i, G)$
  - $tb(G)$: minimum breadth over all tree-decompositions



- Lemma: If a graph G admits a multiplicative tree t-spanner then $tb(G) \leq \lceil t/2 \rceil$ [Dragan+'11]

Known results on Spanners

- Multiplicative tree t-spanner on general graphs
  - $t \geq 4$ is NP-Complete, linear for $t = 1,2$ open for $t = 3$ [Cai+'95]
  - NP-hard for 2-approximation [Liebchen'08]
  - $O(\log n)$-approximation by [Emek+'08] and [Dragan+'11]
- Sparse t-spanner on general graphs
  - $t, m \geq 1$ is NP-complete [Peleg+'89]
  - NP-hard for $o(\log n)$-approximation [Kortsraz'99]
- Additive tree t-spanner on general graphs
  - NP-hard for $o(n)$-approximation [Emek+'08]
- Collective additive tree r-spanners
  - No constant number of trees guarantee $+r$ (r constant) even for outerplanar graphs [Dragan++'06]

From a tree spanner to a sparse spanner

- ∃ tree $(\times\, t)$-spanner $\Longrightarrow$ ∃ $(\times\, t)$-spanner with $O(n \log n)$ edges [Elkin+'05]

From a multiplicative to additive(?)

- $tl(G) = \lambda \implies \exists\ (+\, 2\lambda)$-spanner with $O(\lambda n + n \log n)$ edges
- $tl(G) = \lambda \implies \exists\ (+\, 4\lambda)$-spanner with $O(\lambda n)$ edges

  [Dourisboure+++'07]

- ∃ tree $(\times\, t)$-spanner $\Longrightarrow tl(G) \leq t$ [Dragan+'11]

- ∃ tree $(\times\, t)$-spanner $\Longrightarrow$ ∃ $(+\, 2t)$-spanner with $O(tn + n \log n)$ edges
- ∃ tree $(\times\, t)$-spanner $\Longrightarrow$ ∃ $(+\, 4t)$-spanner with $O(tn)$ edges

From a multiplicative to additive(?)

- $\exists$ tree $(\times\, t)$-spanner $\Longrightarrow$ $\exists$ $(+\, 2t)$-spanner with $O(tn + n\log n)$ edges

- $\exists$ tree $(\times\, t)$-spanner $\Longrightarrow$ $\exists$ $(+\, 4t)$-spanner with $O(tn)$ edges

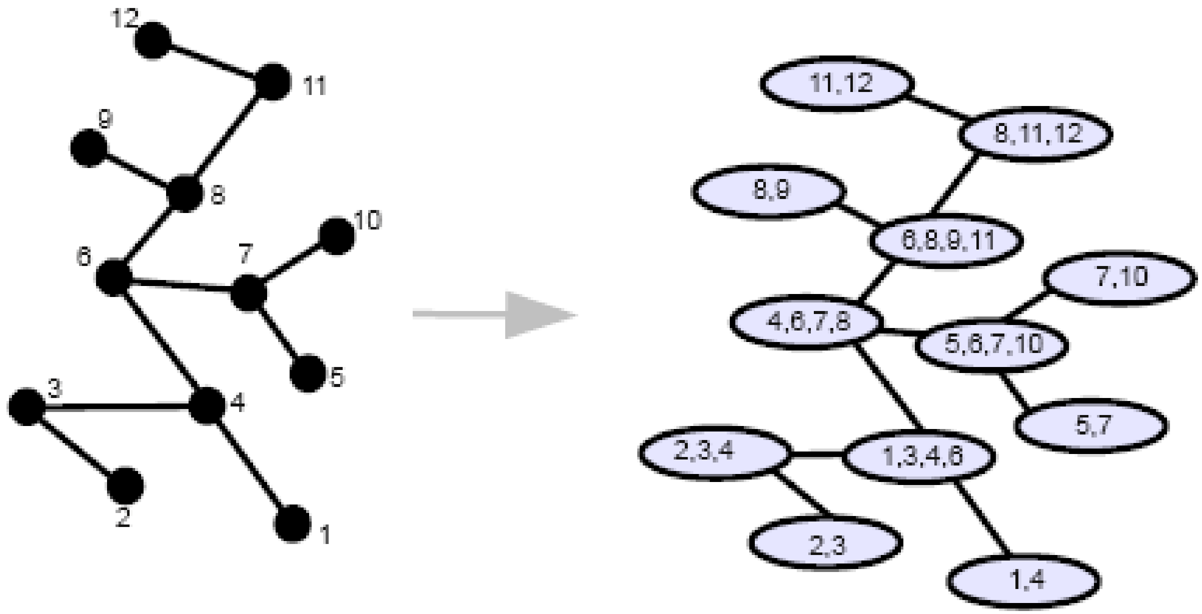
???

- $\exists$ tree $(\times\, t)$-spanner $\Longrightarrow$ $\exists$ $(+\, O(t))$-spanner with $O(n)$ or $O(n\log n)$ edges?
- $\exists$ tree $(\times\, t)$-spanner $\Longrightarrow$ $\exists$ system of $\tilde{O}(1)$ collective tree $(+\, \tilde{O}(t))$-spanners?

- $\exists$ tree $(\times\, t)$-spanner $\Longrightarrow$ $\exists$ tree $(+\, o(n)t)$-spanner is NP-hard [Emek+’08]

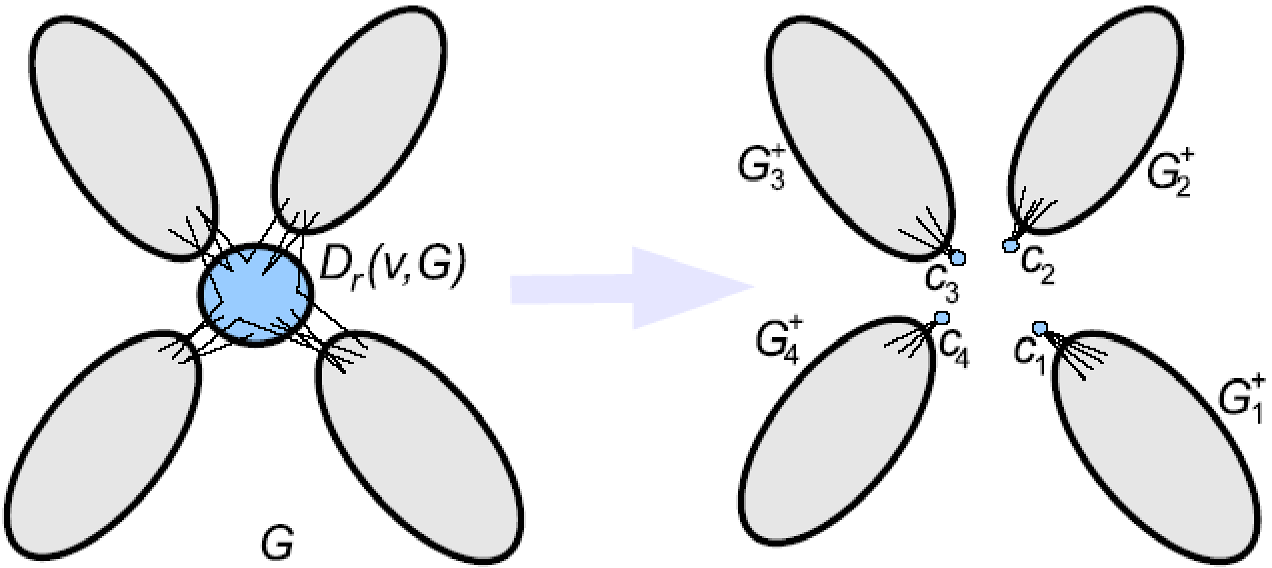
Tree-breadth and multiplicative t-spanner

- Lemma: If a graph G admits a multiplicative tree t-spanner then $tb(G) \leq \lceil t/2 \rceil$. [Dragan+'11]



Hierarchical decomposition of a graph with bounded tree-breadth

- Lemma: Every graph G has a balanced disk-separator $D_r(v, G)$ with $r \leq tb(G)$. It can be found $O(nm)$ time.



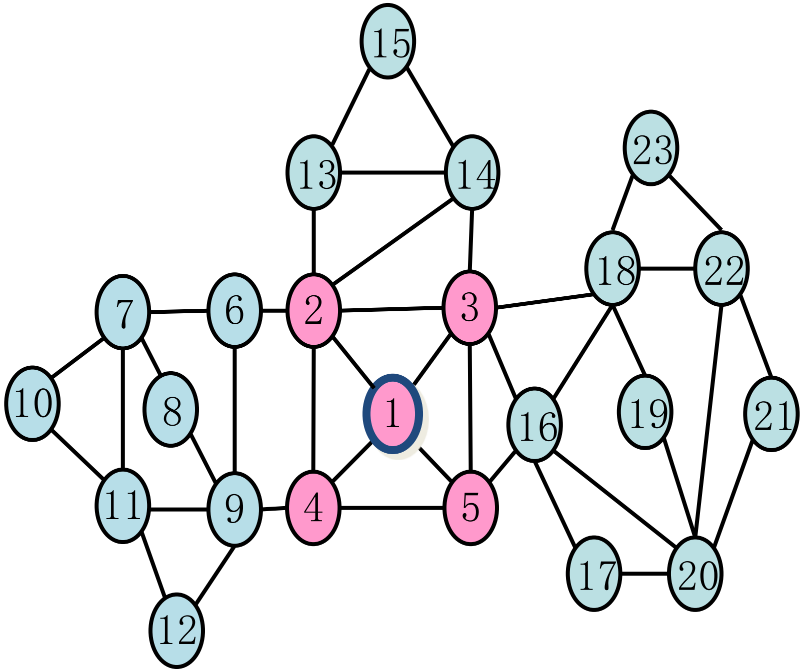
$G_1, \ldots, G_q$ are components of $G \backslash D_r(v, G)$, $G_i^+ \rightarrow (G_i$ with contracted $D_r(v, G))$

- Lemma: $tb(G_i^+) \leq tb(G)$.

Hierarchical decomposition of a graph with bounded tree-breadth

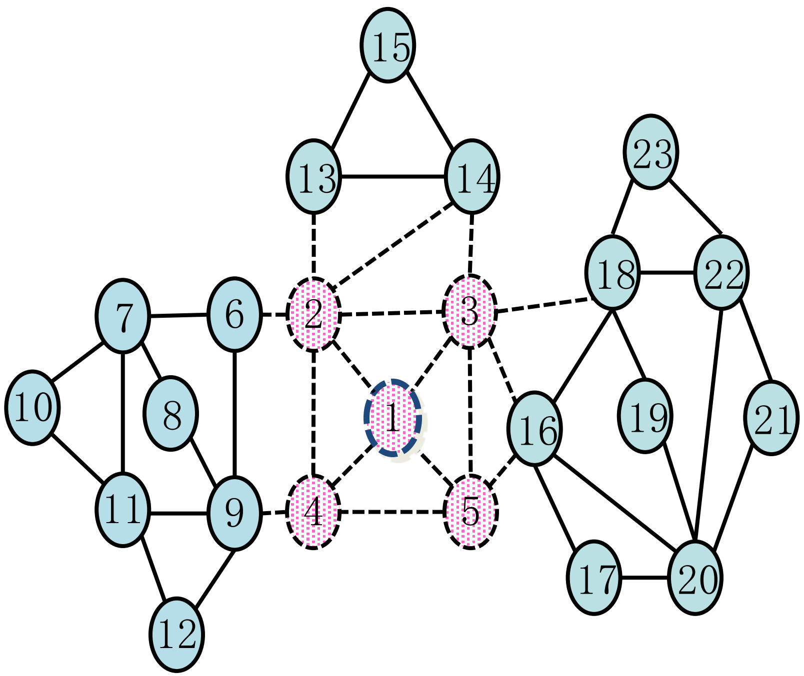
- Find a balanced disk-separator $D_r(v, G)$ with minimum r

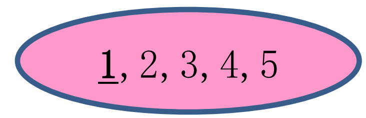
$r \leq tb(G)$



Hierarchical decomposition of a graph with bounded tree-breadth

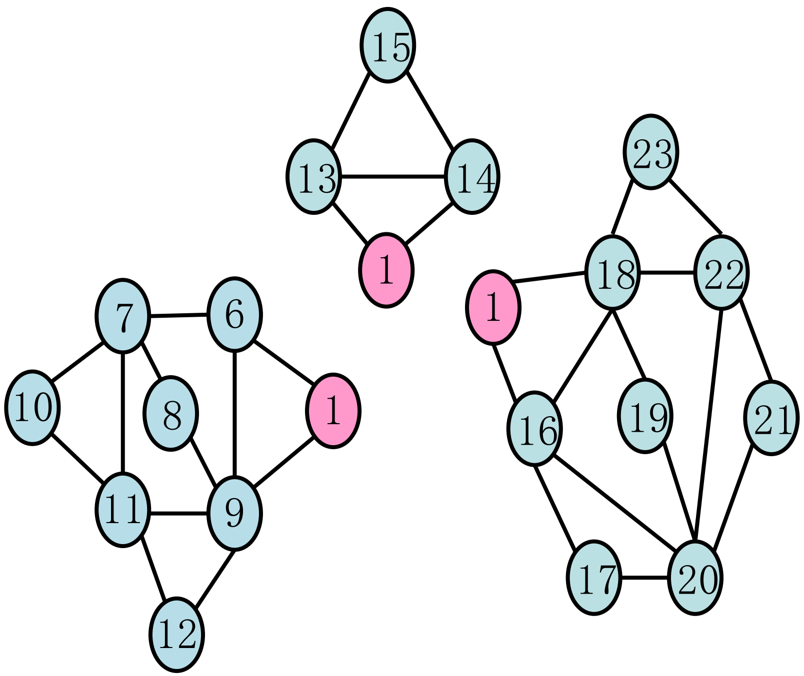
- Use $D_r(v, G)$ as the root of the rooted balanced tree

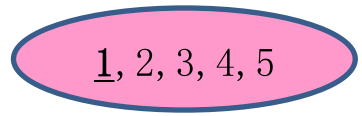




Hierarchical decomposition of a graph with bounded tree-breadth

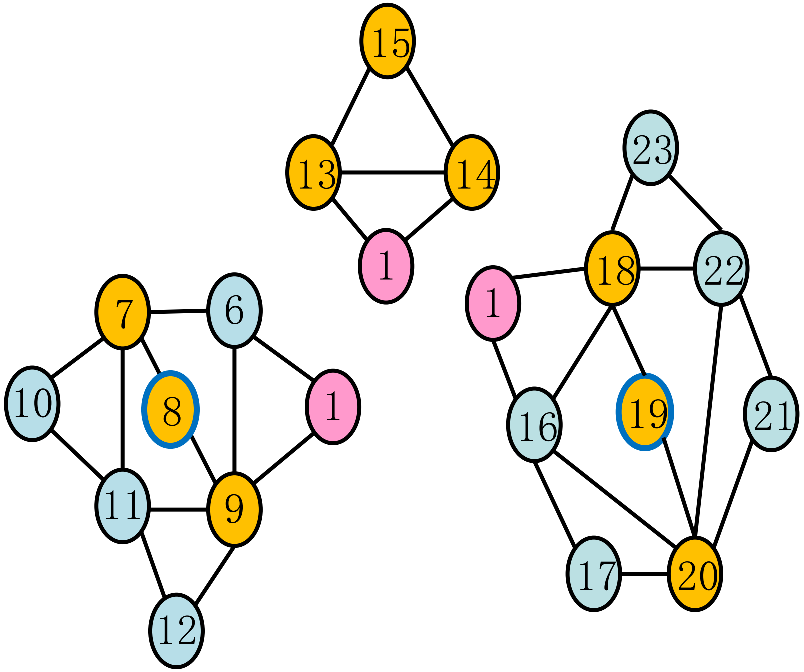
- Build $G_1^+, \ldots, G_q^+$

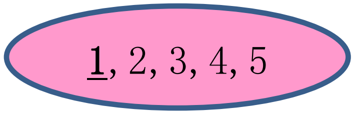




Hierarchical decomposition of a graph with bounded tree-breadth

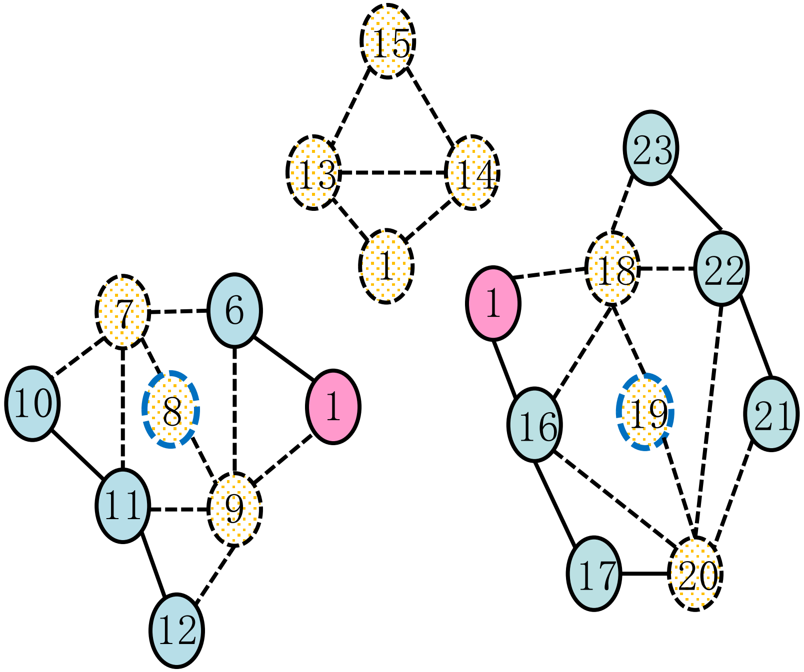
- for each G_i^+, find its balanced disk-separator

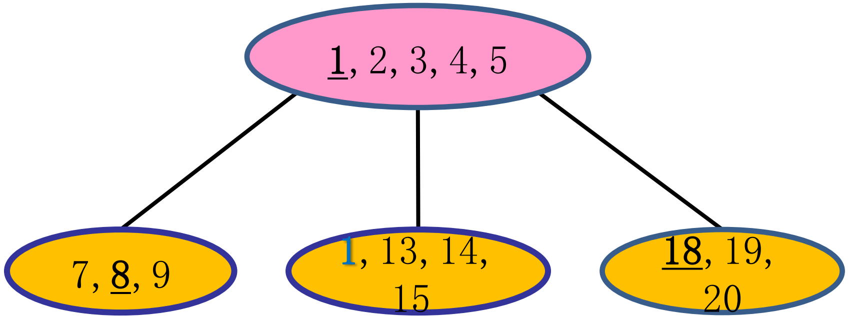




Hierarchical decomposition of a graph with bounded tree-breadth

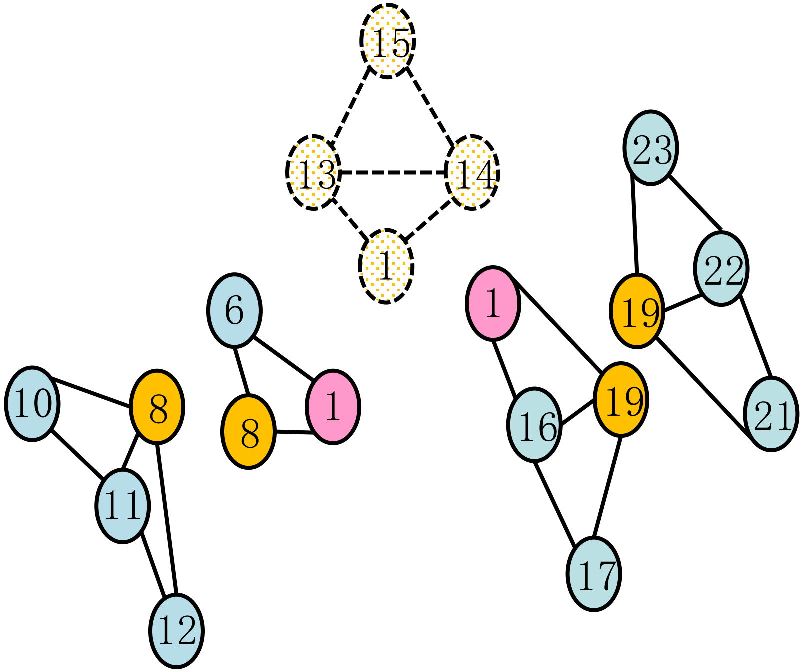
- Use the separators as nodes of the *rooted balanced tree* and let $D_r(v, G)$ be their father

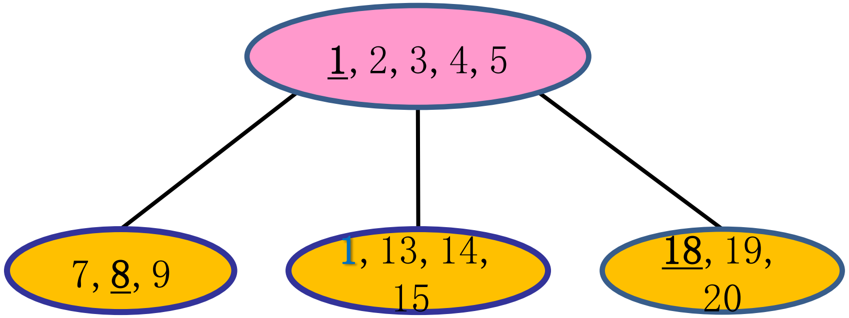




Hierarchical decomposition of a graph with bounded tree-breadth

- Recursively repeat previous procedure until each connected component has at most 5 vertices.





Hierarchical decomposition of a graph with bounded tree-breadth

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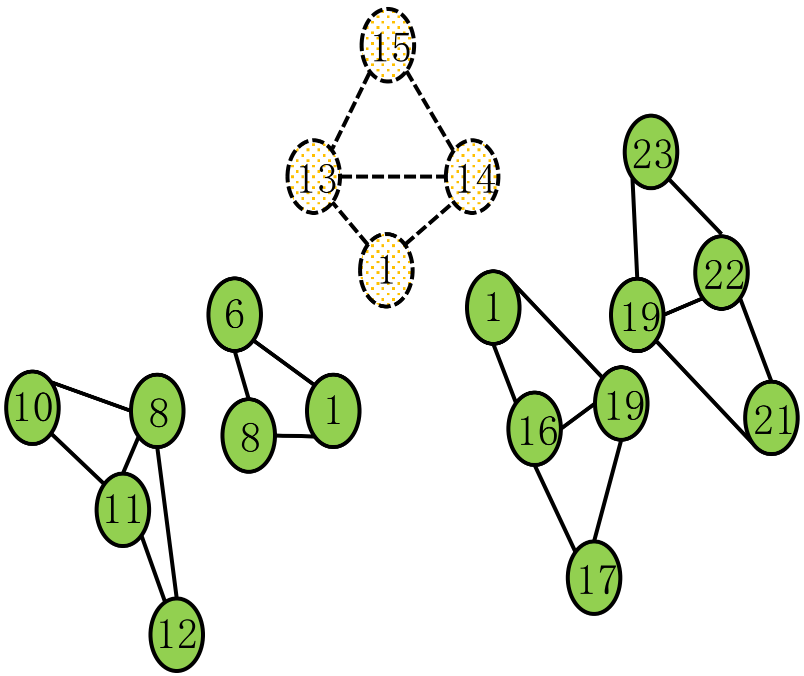
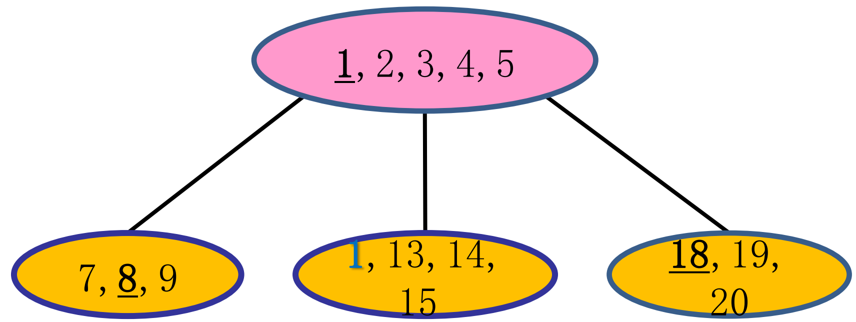
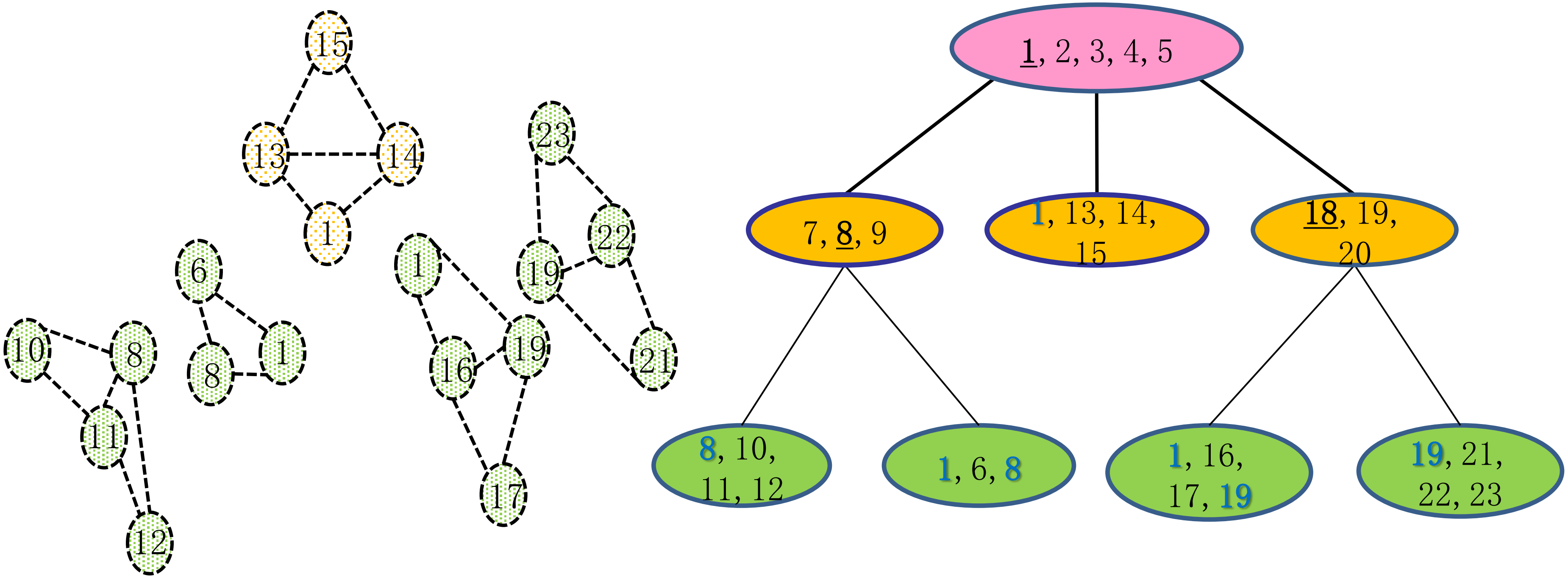




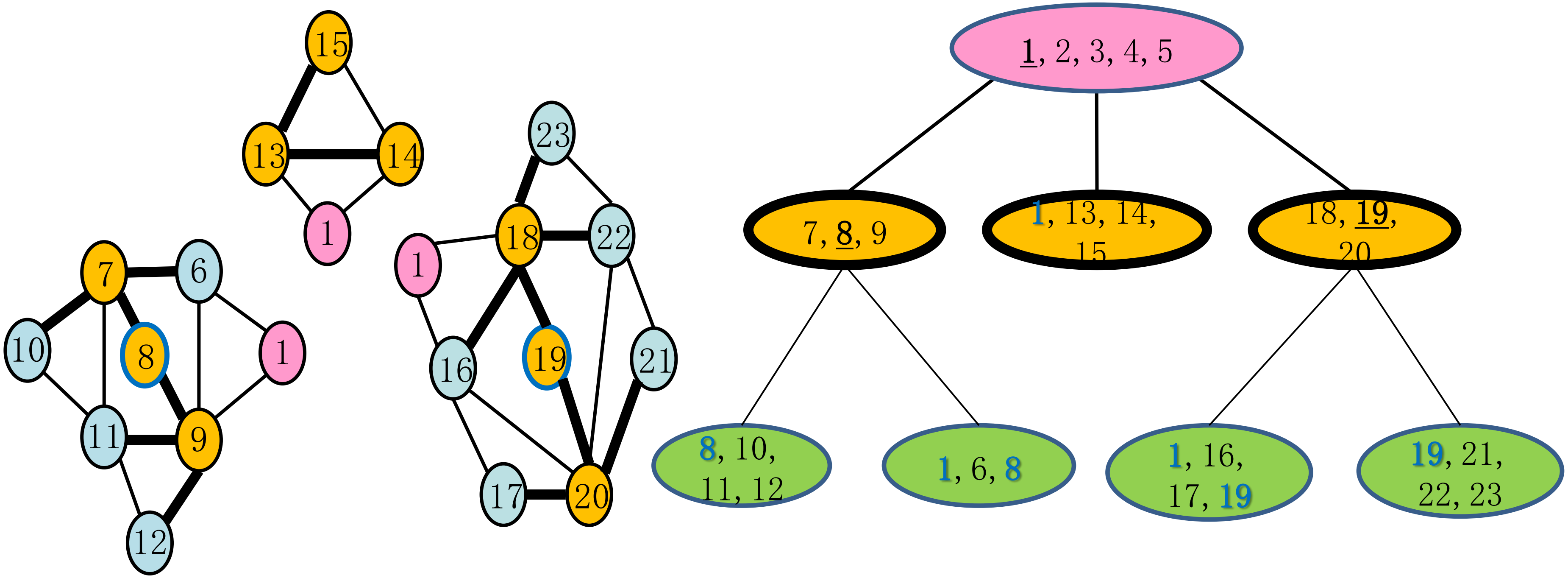
Hierarchical decomposition of a graph with bounded tree-breadth

- Final rooted balanced tree



Constructing Local Spanning Trees

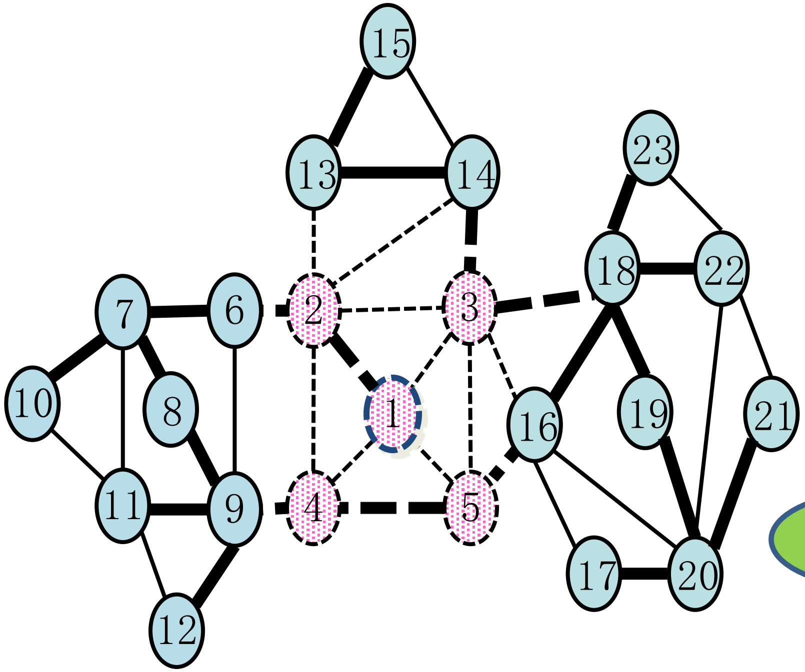
- Construction of local spanning trees of the 2nd layer.

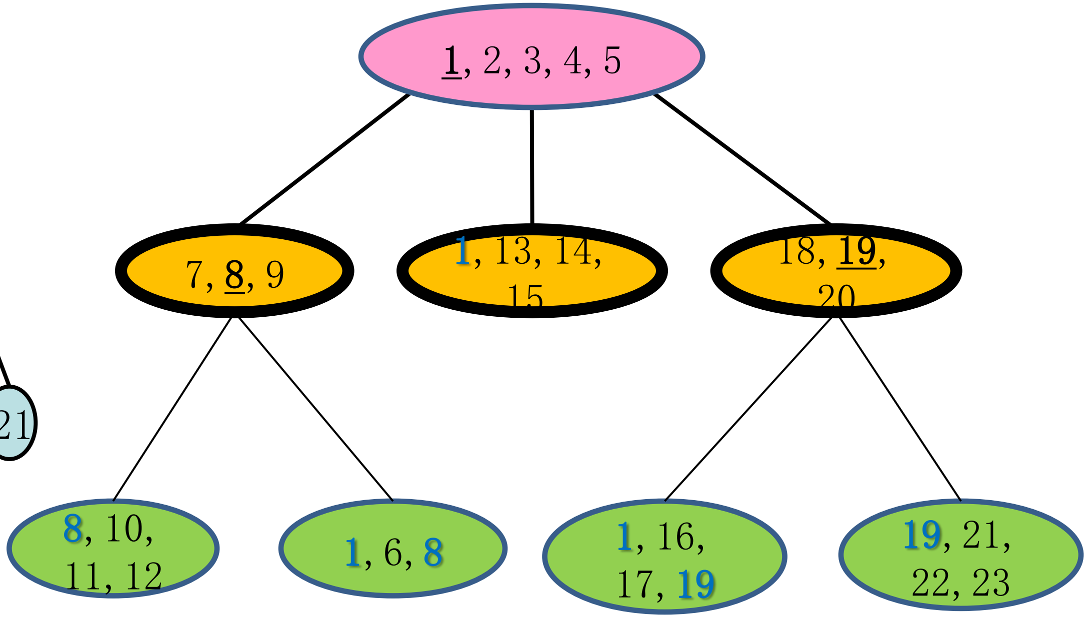


- Local tree is a shortest path-tree centered at the disk center of each node.

Constructing Spanning Tree

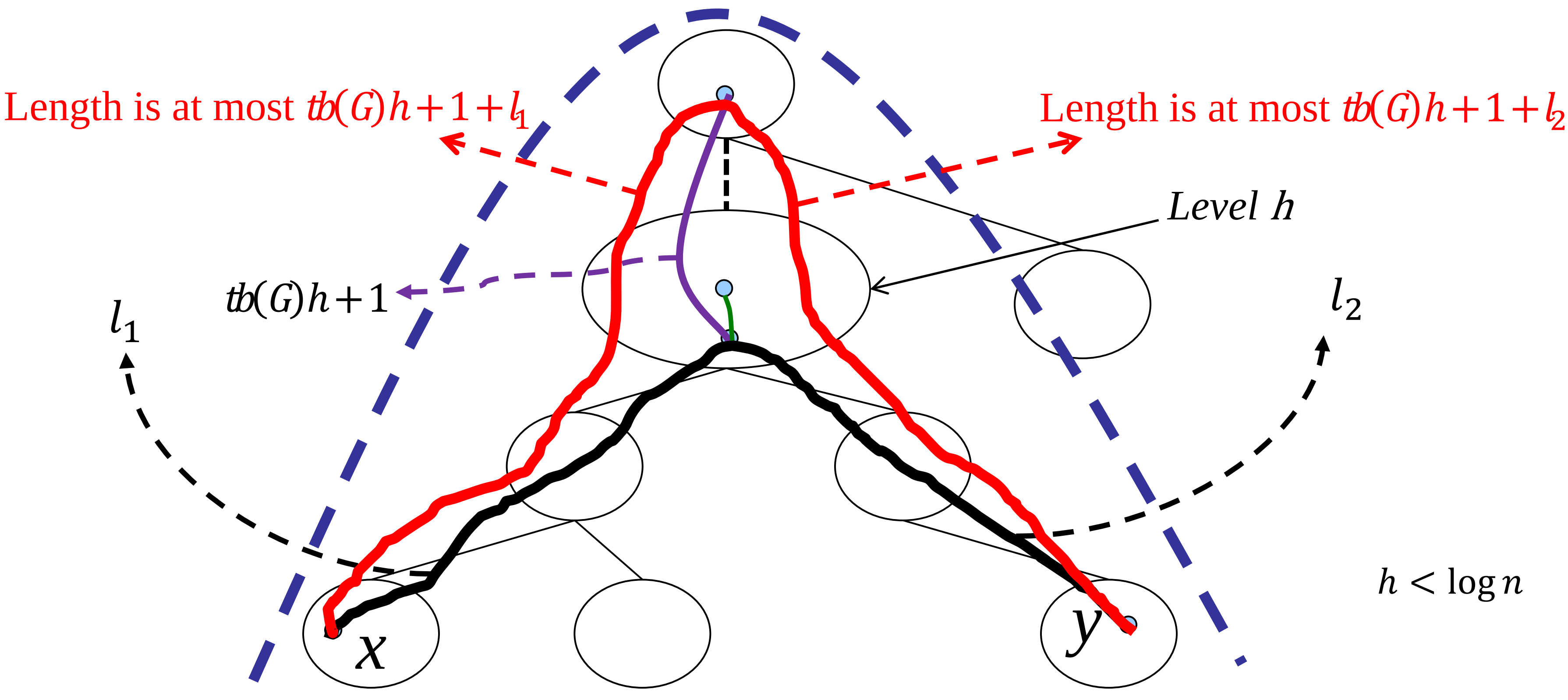
- A spanning tree for the 2nd layer.





Main Result

- Thm: ∃ $\log n$ collective tree $(+\,2tb(G)\log n)$-spanners constructible in polynomial time.



Results

- Thm: ∃ $\log n$ collective tree $(+\,2tb(G)\log n)$-spanners constructible in polynomial time.
- Thm: ∃ $(+\,2tb(G)\log n)$-spanner with at most $\mathrm{n}\log n$ edges constructible in polynomial time.

- Lemma: ∃ tree $(\times\, t)$-spanner $\Longrightarrow$ $tb(G) \leq \lceil t/2 \rceil$. [Dragan+'11]

Consequences to general graphs

- Thm: ∃ tree $(\times\, t)$-spanner $\Longrightarrow$ ∃ system of $\log n$ collective tree $(+\,2\lceil t/2 \rceil \log n)$-spanners constructible in $O(nm\log^2 n)$.
- Thm: ∃ tree $(\times\, t)$-spanner $\Longrightarrow$ ∃ $(+\,2\lceil t/2 \rceil \log n)$-spanner with at most $n\log n$ edges constructible in $O(nm\log^2 n)$.

Earlier results

- $\exists$ tree $(\times\, t)$-spanner $\Longrightarrow$ $\exists$ $(\times\, t)$-spanner with $O(n \log n)$ edges [Elkin+'05]

- $\exists$ tree $(\times\, t)$-spanner $\Longrightarrow$ $\exists$ $(+\,2t)$-spanner with $O(tn + n \log n)$ edges

- $\exists$ tree $(\times\, t)$-spanner $\Longrightarrow$ $\exists$ $(+\,4t)$-spanner with $O(tn)$ edges

- $\exists$ tree $(\times\, t)$-spanner $\Longrightarrow$ $\exists$ tree $(+\,o(n)t)$-spanner is NP-hard [Emek+'08]

Our results

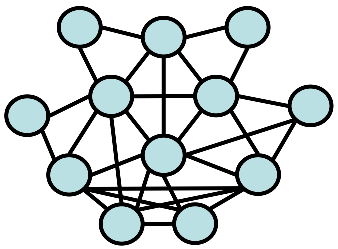
- Thm: $\exists$ tree $(\times\, t)$-spanner $\Longrightarrow$ $\exists$ system of $\log n$ collective tree $(+O\,(t \log n))$-spanners

- Thm: $\exists$ tree $(\times\, t)$-spanner $\Longrightarrow$ $\exists$ $(+\,O(t \log n))$-spanner with at most $n \log n$ edges

Generalized Results

- k-TREE-WIDTH t-SPANNER PROBLEM:

  Given unweighted undirected graph $G = (V, E)$ and integers k, t.
  Does G admit a t-spanner $H = (V, E')$ of tree-width at most k?



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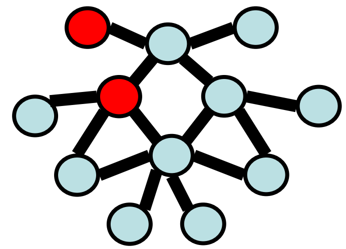
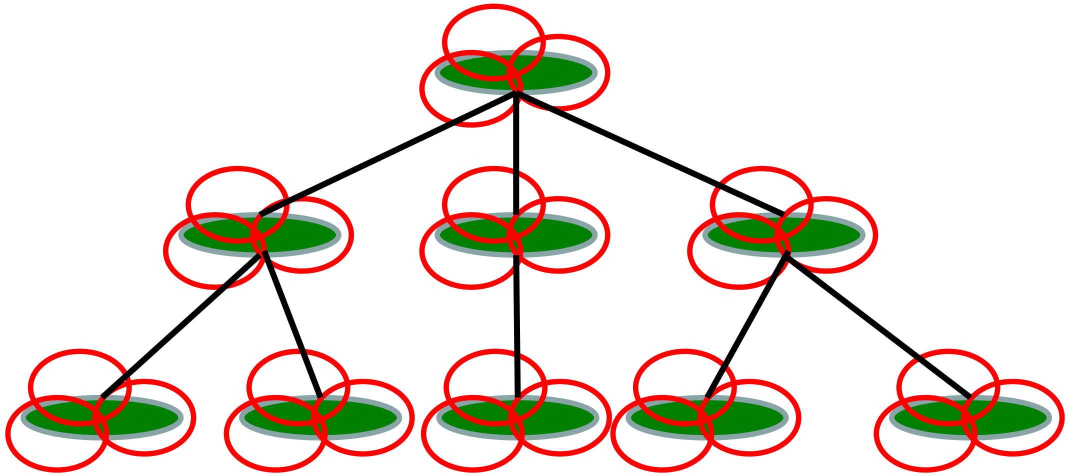


2-spanner of tree-width 2

- Thm: $\exists$ $(\times\, t)$-spanner with $tw = k \Longrightarrow \exists$ system of $k(1 + \log n)$ collective tree $(+\, 2\lceil t/2 \rceil(1 + \log n))$-spanners constructible in polynomial time for every fixed k.
- Thm: $\exists$ $(\times\, t)$-spanner with $tw = k \Longrightarrow \exists$ $(+\, 2\lceil t/2 \rceil(1 + \log n))$-spanner with at most $O(kn \log n)$ edges constructible in polynomial time for every fixed k.

k-Tree-Breadth and t-Spanner with Bounded Tree-Width

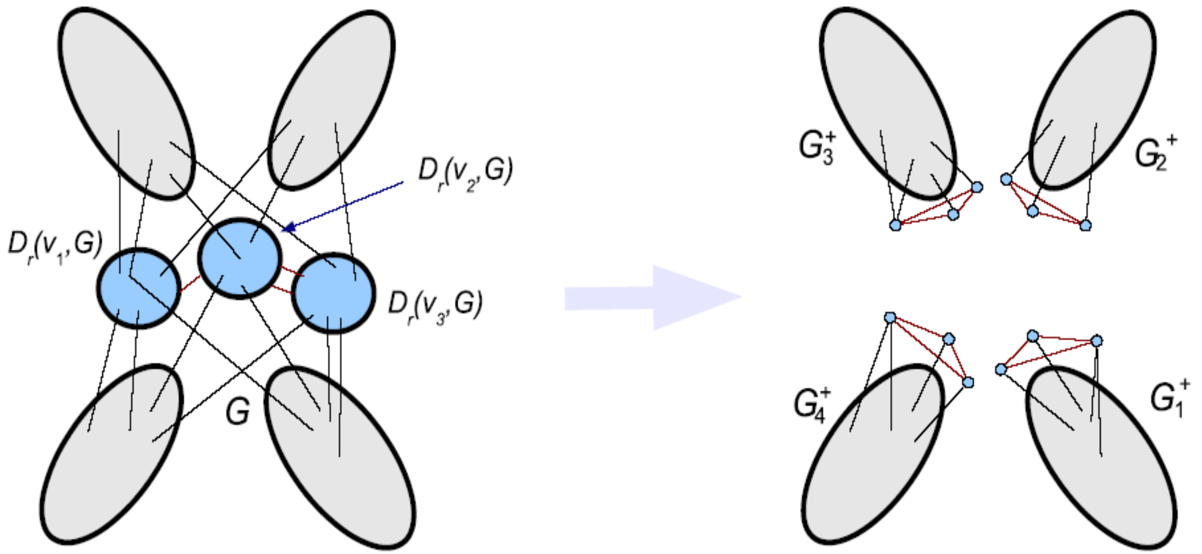
- k-tree-breadth $tb_k(G)$:
  - k-breadth: minimum r such that $\forall i \in I,\ \exists v_1, \ldots, v_k$ with $X_i \subseteq \bigcup_{j=1}^{k} D_r(v_j, G)$
  - $tb_k(G)$: minimum k-breadth over all tree-decompositions



- Lemma: If a graph G admits a t-spanner with tree-width $k-1$, then $tb_k(G) \leq \lceil t/2 \rceil$.

Collective Additive Tree Spanners of Graphs with Bounded k-Tree-Breadth

G has D_r^k-separator with $r \leq tb_k(G)$



$tb_k(G_i^+) \leq tb_k(G)$

Constructions of collective additive tree spanner similar for that of graphs with bounded tree-breadth

Results for Collective Additive Tree Spanners of Graphs with Bounded k-Tree-Breadth

- Thm: ∃ a system of $k(1 + \log n)$ collective tree $(+\,(2tb_k(G)(1 + \log n)))$-spanners constructible in polynomial time for every fixed k.
- Thm: ∃ $(+\,(2tb_k(G)(1 + \log n)))$-spanner with at most $O(kn \log n)$ edges constructible in polynomial time for every fixed k.
- Thm: ∃ $(\times\, t)$-spanner with $tw = k \Longrightarrow$ ∃ system of $k(1 + \log n)$ collective tree $(+\,2\lceil t/2 \rceil(1 + \log n))$-spanners constructible in polynomial time for every fixed k.
- Thm: ∃ $(\times\, t)$-spanner with $tw = k \Longrightarrow$ ∃ $(+\,2\lceil t/2 \rceil(1 + \log n))$-spanner with at most $O(kn \log n)$ edges constructible in polynomial time for every fixed k.

Challenges

- G admitting a multiplicative tree t-spanner $\Longrightarrow$ $O(1)$ collective additive tree $O(t)$-spanners of G? $[logn, +O(tlogn)]$
- G admitting a multiplicative t-spanner with tree-width k $\Longrightarrow$ $O(k)$ collective additive tree $O(kt)$-spanners of G? $[O(k \log n), O(t \log n)]$
- G admitting μ collective tree t-spanners $\Longrightarrow$ $\alpha(\mu, n)$ collective tree $\beta(t, n)$-spanners, where $\alpha(\mu, n)$ is $O(\mu)$ (or $O(\mu \log n)$) and $\beta(t, n)$ is $O(t)$ (or $O(t \log n)$)?
- G admitting a t-spanner of tree-width k $\Longrightarrow$ $(O(k \log n)t)$-spanner with tree-width at most k?

Thank You