Computing the Hull Number of Chordal and Distance-Hereditary Graphs

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Betweenness Relations

A **betweenness** on a ground set V is a ternary relation B such that

$$B(x, z, y) \iff B(y, z, x)$$

$B(x, z, y)$ is pronounced "z is between x and y".

$X \subseteq V$ is said **convex** if $X = \bigcup_{x,y \in X} \{z \mid B(x, z, y)\}$.

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Ex. $B = \{(a, c, b), (b, c, a), (b, a, d), (d, a, b), (a, c, d), (d, c, a)\}$.
$\{a, b\}$ is not convex, but $\{a, b, c, d\}$ is.

A **convex hull** of X is $\min\{Z \text{ convex} \mid X \subseteq Z\}$.

X is a **hull set** if its convex hull is V.

Betweenness Relations

Betweenness Relations appear in the literature in several works.

- Karl Menger in his works in metric spaces.
  Untersuchungen über allgemeine Metrik, 1928
- Hans Reichenbach in probability theory (events causality).
  The direction of time, 1956
- Robert J. Bumcrot in geometry and lattices.
  Betweenness geometry in lattices, 1964
- Vašek Chvátal in Antimatroids and convexity spaces.
  Antimatroids, Betweenness, Convexity, 2009
- Others in Graph Theory.
  See Survey by Pelayo, 2004

Geodesic Betweenness in Graphs

The **shortest path** between x and y is a chordless path of minimum length between x and y.

The $\mathcal{SP}_G$ betweenness on V_G is the one where $\mathcal{SP}_G(x, z, y)$ holds if z is in a shortest path between x and y.

The **hull number** of G is $\min\{|X| \mid X \text{ hull set}\}$.

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Interest. Characterisation of graphs by means of their hull number and applications of geometry to graph theory.

Computational complexity. NP-complete in all (bipartite) graphs.
Left open for several graph classes (DH, planar, chordal).

Results

Theorem

- *One can compute in time $O(m+n)$ a hull set of minimum size in distance-hereditary graphs.*
- *One can compute in time $O(n^3)$ a hull set of minimum size in chordal graphs.*

A graph is **distance-hereditary** if distances are preserved in connected induced subgraphs.

A graph is **chordal** if it does not contain cycles of length ≥ 4.

Plan

Chordal Graphs

A vertex is **simplicial** if its neighbourhood is a clique.

A **perfect elimination ordering** of G is an ordering $(x_1, \ldots, x_n)$ such that x_i is simplicial in $G[\{x_i, \ldots, x_n\}]$.

We borrow ideas from Database Theory and use the following results by Dirac'61, Fulkerson-Gross'65 and Tarjan-Lueker'76.

Theorem 1

(i) Every chordal graph has at least two simplicial vertices.

(ii) G is chordal iff it has a perfect elimination ordering.

(iii) A perfect elimination ordering of a chordal graph can be computed in time $O(n + m)$.

Functional Dependencies

A **functional dependency** on a ground set V is a pair (X, y), written $X \to y$, with X the **premise** and y the **conclusion**.

An **implicational system** is set of functional dependencies on V.

F is **closed** if $y \in F$ whenever $X \subseteq F$, for all $X \to y \in \Sigma$.

The **closure** of X, $\Sigma(X)$, is the smallest closed set containing X.

A **key** is an inclusionwise minimal set X such that $\Sigma(X) = V$.

Betweenness Relations as Implicational Systems

A Betweenness relation is an implicational system with premises of size 2.

To every graph G, we associate the implicational system

$$\Sigma_G := \bigcup_{x,y \in V} \{xy \to z \mid \mathcal{SP}_G(x,z,y) \text{ holds}\}.$$

Fact 1

K is a minimum key of Σ_G iff K is a minimum hull set of G.

The Algorithm

A vertex x is an **extreme point** in Σ if x is not a conclusion.

Ex. Simplicial vertices are extreme points in Σ_G.

Algorithm

1. Construct $\Sigma := \Sigma_G$ and take a perfect elimination ordering $(x_1, \ldots, x_n)$.
2. For each i, decide whether to put x_i in the key and let $\Sigma := \Sigma \setminus x_i$.
3. The remaining implicational system, if exists, is with premises of size 1. Compute a key and add it to the already computed one.
4. Return the computed key.

Correctness

$$\Sigma' := \Sigma \setminus x_1 \setminus \ldots \setminus x_i.$$

Lemma 1

If x_{i+1} is an extreme point in Σ', then any key of Σ' is of the form $K \cup \{x_{i+1}\}$ where K is a key of $\Sigma' \setminus x_{i+1}$ defined as

$$\{zy \to t \in \Sigma' \mid z, t, y \neq \Sigma'(\{x_{i+1}\})\} \quad \bigcup$$

$$\{y \to z \mid yx \to z \in \Sigma' \text{ and } x \in \Sigma'(\{x_{i+1}\}),\ y, z \notin \Sigma'(\{x_{i+1}\})\}.$$

Remove from $\Sigma \setminus x_1 \setminus \ldots \setminus x_i$ all those vertices that can be obtained from x_{i+1} to get $\Sigma \setminus x_1 \setminus \ldots \setminus x_i \setminus x_{i+1}$.

Correctness

$$\Sigma' := \Sigma \setminus x_1 \setminus \ldots \setminus x_i.$$

Lemma 2

If x_{i+1} is not an extreme point in Σ', then it appears as a conclusion only in functional dependencies with premises of size 1. Define $\Sigma' \setminus x_{i+1}$ as

$$\Sigma' \setminus \{zx_{i+1} \to y \in \Sigma'\}) \cup \left(\{tz \to y \mid zx_{i+1} \to y, t \to x_{i+1} \in \Sigma'\}\right)$$

A minimum key in $\Sigma' \setminus x_{i+1}$ is a minimum key in Σ'. Conversely, to any minimum key in Σ', one can associate a minimum key in $\Sigma' \setminus x_{i+1}$.

We cannot decide whether to put x_{i+1} in a key, however we can replace it safely from $\Sigma \setminus x_1 \setminus \ldots \setminus x_i$.

Time Complexity

Proposition 1

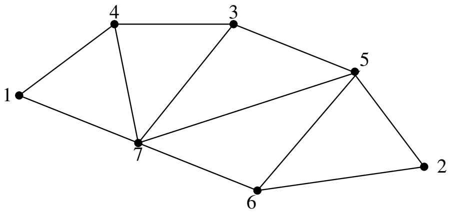
For every graph G, Σ_G can be computed in time at most $O(n^3)$.

Proposition 2

If Σ is an implicational system on V with premises of size 1, then a minimum key of Σ can be computed in time $O(|V| + |\Sigma|)$.

Example

A chordal graph G and its associated implicational system.



$12 \to 567$

$13 \to 47$

$15 \to 6$

$16 \to 7$

$23 \to 5$

$24 \to 3567$

$27 \to 56$

$36 \to 57$

$45 \to 37$

$46 \to 7$

Example

1 is an extreme point in Σ and set $K := \{1\}$

Σ	$\Sigma \setminus 1$
$12 \to 567$	$2 \to 567$
$13 \to 47$	$3 \to 47$
$15 \to 6$	$5 \to 6$
$16 \to 7$	$6 \to 7$
$23 \to 5$	$23 \to 5$
$24 \to 3567$	$24 \to 3567$
$27 \to 56$	$27 \to 56$
$36 \to 57$	$36 \to 57$
$45 \to 37$	$45 \to 37$
$46 \to 7$	$46 \to 7$

Example

2 is an extreme point in $\Sigma \setminus 1$ and set $K := \{1, 2\}$

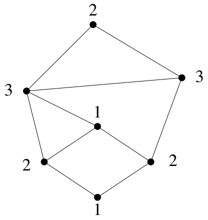
Σ	$\Sigma \setminus 1$	$\Sigma \setminus 1 \setminus 2$
$12 \to 567$	$2 \to 567$	
$13 \to 47$	$3 \to 47$	
$15 \to 6$	$5 \to 6$	
$16 \to 7$	$6 \to 7$	
$23 \to 5$	$23 \to 5$	$3 \to 4$
$24 \to 3567$	$24 \to 3567$	$4 \to 3$
$27 \to 56$	$27 \to 56$	
$36 \to 57$	$36 \to 57$	
$45 \to 37$	$45 \to 37$	
$46 \to 7$	$46 \to 7$	

Plan

Clique-Width

A complexity measure based on Graph Grammars.

A k-**graph** = each vertex has *exactly one* colour in $\{1, \ldots, k\}$.



3-graphe

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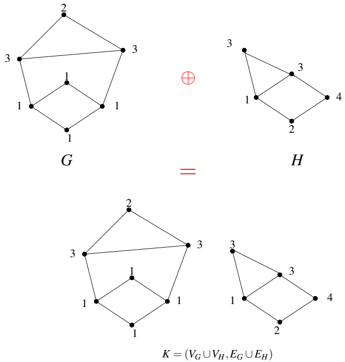
$G \oplus H$ = disjoint union of k-graphs.



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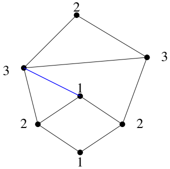
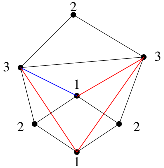


G



$add_{1,3}(G)$

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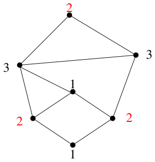
$add_{i,j}(G)$ = addition of edges between i-vertices and j-vertices.

$ren_{i \to j}(G)$ = recolour i-vertices into j-vertices.



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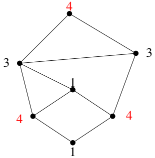

$ren_{2 \to 4}(G)$

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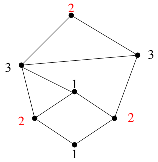
$\mathbf{i}$ = a graph with one vertex coloured i.



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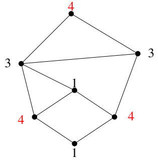
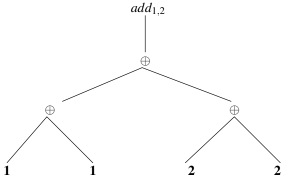


$ren_{2 \to 4}(G)$

Clique-Width

- $F_k = \{\oplus, add_{i,j}, ren_{i \to j} \mid i,j \in [k]\}$.
- $C_k = \{\mathbf{i} \mid i \in [k]\}$

A term t in $T(F_k, C_k)$ defines a graph $val(t)$.



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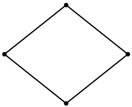

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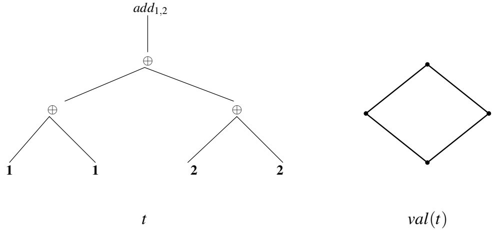

$$\text{cwd}(G) := \min\{k \mid G = val(t), t \in T(F_k, C_k)\}$$

Monadic Second-Order Logic

A k-graph is the relational structure $\langle V_G, edg_G, (p_{iG})_{i\in[k]}\rangle$.

Atomic Formulas. $x \in X$, $edg(x,y)$, $p_i(x)$, $x = y$.

MSO formulas. Boolean combinations and element/set quantifications.

Ex. $\forall X(x \in X \ \wedge \ \forall z, t(z \in X \ \wedge \ edg(z,t) \implies t \in X) \implies y \in X)$.

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MSO optimisation. Find a tuple $(Z_1, \ldots, Z_q)$ of $(2^{V_G})^q$ such that

$$\sum_{1\leq i\leq q} |Z_i| = opt\left\{ \sum_{1\leq i\leq q} \left|W_j\right| \mid G \models \varphi(W_1, \ldots, W_q) \right\}.$$

MSO and Clique-Width

Theorem 2 (Courcelle,Makowski,Rotics'00 and Oum'05)

Every MSO optimisation problem can be solved in time $O(f(k) \cdot n^3)$ in graphs of clique-width at most k. If clique-width expression is given, it can be solved in time $O(g(k) \cdot n)$.

MSO definability of Hull Set

Proposition 3

If there exists an MSO formula $\varphi(x, z, y)$ stating that z is in a shortest path between x and y, then there exists an MSO formula stating that X is a hull set.

$$Cl(X) \equiv \forall x, y(x \in X \ \wedge \ y \in X \implies \neg\exists z(\varphi(x, z, y))),$$

$$CH(X, Y) \equiv Cl(Y) \wedge X \subseteq Y \wedge \forall Z(X \subseteq Z \wedge Z \subseteq Y \implies \neg Cl(Z))$$

$$HullSet(X) \equiv \forall Z\big(Z \subsetneq V \implies \neg CH(X, Z)\big)$$

Hull Number of DH Graphs

G is distance-hereditary iff chordless paths are shortest paths.

There exists an MSO formula stating that z is in a chordless path between x and y in a graph.

Distance-Hereditary graphs have clique-width at most 3 and clique-width expressions can be computed in time $O(n+m)$.

Combine Theorem 2 and Proposition 3.

Plan

Concluding Remarks

Conjecture. NP-complete in planar graphs, but polynomial in bounded degree and clique-width bounded graphs.

Techniques for DH graphs can be used for other betweenness relations (triangle paths, monophonic paths, etc.) to compute a minimum hull set in clique-width bounded graphs.

Betweenness relations give **dependence graphs** and allow to MSO define any betweenness relation. Characterise those of bounded clique-width.

Dichotomy. Find a sharp line between tractable and intractable cases. Can the lattice structure of betweenness relations can help ?

Thank you ! !