Online and Quasi-online Colorings of Wedges and Intervals

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Summary of problems of interest

Investigated objects

- primal: coloring *points* on a line with respect to *intervals*
- dual: coloring *intervals* on a line with respect to *points*
- primal: coloring *points* in the plane with respect to *wedges*
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From offline to online

- offline
- quasi-online
- semi-online
- online

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Applications

- resource allocation (minimize number of CPU's to run several jobs)
- cover-decomposability problems → sensor networks
- conflict-free colorings → frequency assignment

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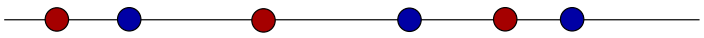

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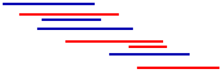

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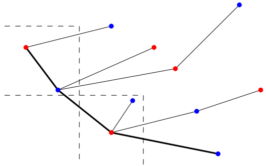

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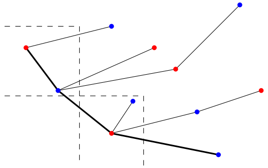

Dual statement (equivalent):

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Wedges are **2-colorable** *such that any point that is contained in at least* **two wedges** *is contained in a red and blue wedge as well.*

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Online

In online coloring, the set of objects to be colored is not known beforehand; objects come to be colored one-by-one and a valid coloring must be maintained at all times.

Points wrt. intervals quasi-online

Folklore

Points are **3-colorable** quasi-online such that any interval that contains at least **2 points** is properly colored, i.e.,contains a red and blue point as well. This is best possible.

We will see the proof later.

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Theorem (K, 2007)

Points are **2-colorable** *quasi-online such that any interval that contains at least* **4 points** *is properly colored, i.e.,contains a red and blue point as well.*

Intervals quasi-online

Theorem (K, 2007)

-Intervals are **2-colorable** *such that any point that is contained in at least* **3 intervals** *is contained in a red and blue wedge as well.*
-Intervals are **3-colorable** *such that any point that is contained in at least* **2 intervals** *is contained in a red and blue wedge as well.*
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Theorem (KLP)

-We give simpler proofs for these statements. We also improve the running times:
-such colorings can be found in $O(n \log n)$ *steps.*

Wedges quasi-online

Theorem (K, Pálvölgyi, 2011)

Points are **2-colorable** *quasi-online such that any wedge that contains at least* **12 points** *is properly colored, i.e., contains a red and blue point as well.*

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Theorem (Cardinal, Korman, 2012)

Points are **4-colorable** *such that any wedge that contains at least* **2 points** *contains a red and blue point as well.*

Dual statement is true again. Also, they prove 4-colorability in a much more general setting.

Points wrt. intervals online

Folklore

Points are **3-colorable** online such that any interval that contains at least **2 points** is properly colored, i.e.,contains a red and blue point as well. This is best possible.

Proof.

In each step for the new point we need to choose a color different from the colors of its left and right neighbor. With 3 colors this is always possible. It is also easy to see that 2 colors are not enough. □

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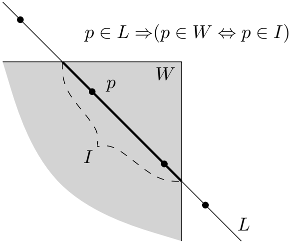
The interval coloring problem is equivalent to a restricted case of the wedges coloring problem, where we care only about the points on the line L defined by $y = -x$. □



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There is a method to color online N points in the plane using $\Theta(\log N/k)$ colors such that all monochromatic wedges have size strictly less than k.

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Corollary

There is a method to online color N intervals in $\mathbb{R}$ using $\Theta(\log N/k)$ colors such that for every point x, contained in at least k intervals, there exist two intervals containing x of different colors.

Wedges online

If the number of colors, c, is fixed, we have:

Theorem (KLP)

- *There exists a method of placing* k^2 *points such that any online 3-coloring of these points produces a monochrom. wedge of size at least* k.
- *There exists a method of online 3-coloring* $k^2 - 1$ *points such that all monochromatic wedges have size less than* k.

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For $c \geq 4$ we can online color with c colors any set of $N = O(1.22074^{ck})$ points such that throughout the process there is no monochrom. wedge of size k.

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Lemma

There exists a method to give $N = 2^n - 1$ points in a sequence such that for any online-coloring method using c colors there will be c monochromatic wedges, $W_1, W_2, \ldots, W_c$, and nonnegative integers $x_1, \ldots, x_c$ such that for each i, the wedge W_i contains exactly x_i points colored with color i and $\sum x_i \geq n+1$ if $n \geq 2$.

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Remark

The same construction can be interpreted with intervals on a line as well.

Lower bound construction

Lemma. There exists a method to give $N = 2^n - 1$ points in a sequence such that for any online-coloring method using c colors **at some point** there will be c monochromatic wedges, $W_1, W_2, \ldots, W_c$, such that the wedge W_i contains exactly x_i points colored with color i and $\sum x_i \geq n + 1$.

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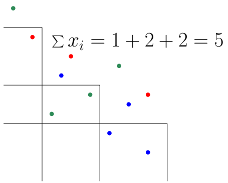
Proof. By induction on n we can place at most $2^{n-1} - 1$ points such that there exist $W_1, W_2, \ldots, W_c$ where the wedge W_i contains x_i points with color i and $\sum x_i = n$.



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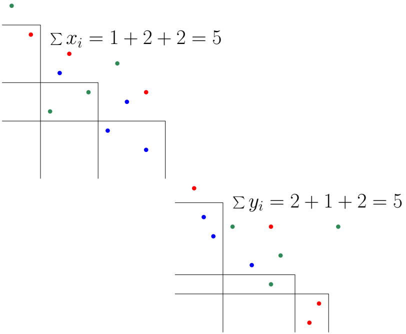
We repeat the procedure south-east to these points, adding at most $2^{n-1} - 1$ more points we get wedges $V_1, V_2, \ldots, V_c$ such that V_i contains y_i points with color i and $\sum y_i = n$.



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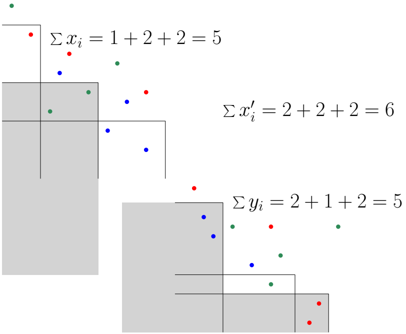
If there exists $x_i \neq y_i$ then at least one $x_j > y_j$ and the wedges $V_1, V_2, \ldots, W_j, V_{j+1}, \ldots V_c$ have the properties of the lemma.



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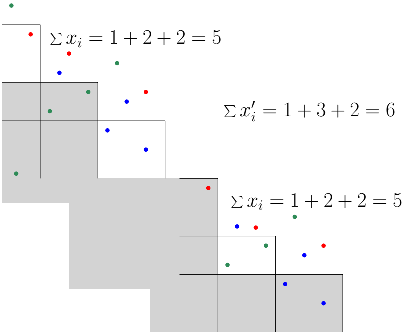
If all $x_i = y_i$ then we add one point south-west from the first set of points. If it is colored j then we get a wedge $|W'_j| = |W_j| + 1$, and the wedges $V_1, V_2, \dots, W'_j, V_{j+1}, \dots V_c$ have the properties of the lemma.



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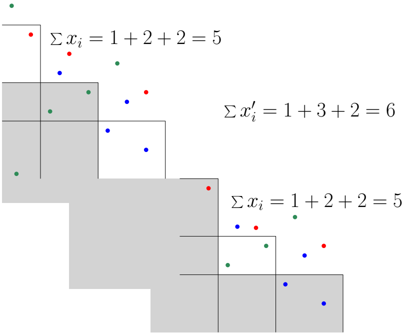
We used at most $2(2^n - 1) + 1 = 2^n - 1$ points indeed.



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However, after we manage to define our preference order for the x_i's, the coloring algorithm is a simple greedy algorithm according to these preferences.
The proof that it gives a valid coloring comes from the fact that we followed our carefully chosen preference order.

Further research

Improve the bounds $N = \Omega(1.46557^{ck})$ and $N = O(2^{ck})$ for which we can *online color* N points with c colors such that there is no monochrom. wedge of size k.

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- for wedges (both primal and dual) $k = 2^{O(2^l)}$ is the best we know (K, Pálvölgyi, 2012); improve this to polynomial or even linear in l.