Algorithms and Almost Tight Results for 3-Colorability of Small Diameter Graphs

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The k-coloring problem

Problem (k-coloring problem)

Given a graph G, can we assign k colors to its vertices such that neighboring vertices receive different colors?

- The k-coloring problem is:
  - NP-complete for $k \geq 3$
  - polynomially solvable for $k = 2$

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- The k-coloring problem is:
  - NP-complete for $k \geq 3$
  - polynomially solvable for $k = 2$
- The 3-coloring problem is NP-complete, even if the given graph is:
  - the line graph of another graph [Holyer, 1981]
  - triangle-free with max. degree at most 4 [Maffray, Preissmann, 1996]
  - planar with max. degree at most 4 [Garey, Johnson, 1979]

  but it becomes polynomial, if the given graph is:
  - perfect [Grötschel et al., 1984]
  - a graph of bounded treewidth [Courcelle, 1990]
  - P_5-free graph [Hoàng et al., 2010]
  - AT-free graph [Stacho, 2012]

The 3-coloring problem

- A central graph parameter: the distance $d(u, v)$ between vertices u, v

Definition

The diameter of a graph G is $diam(G) = \max\{d(u, v) : u, v \in V\}$.

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- Standard results:

Theorem

3-coloring is NP-complete for graphs G with $diam(G) \leq 4$.

Theorem

4-coloring is NP-complete for graphs G with $diam(G) \leq 2$.

Proof.

Reduce from 3-coloring on arbitrary graphs: add a universal vertex. □

The 3-coloring problem

Two longstanding open problems

Is 3-coloring tractable on graphs G with $diam(G) \leq 2$?

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- These problems are also open if the given graph is triangle-free.

Observation

G is *triangle-free* with $diam(G) \leq 2 \iff G$ is *maximal triangle-free*.

The 3-coloring problem

Two longstanding open problems

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- These problems are also open if the given graph is triangle-free.

Observation

G *is triangle-free with* $diam(G) \leq 2 \iff G$ *is maximal triangle-free.*

Other known results do not help with 3-coloring of $diam(G) \leq 2$ graphs:

- it is NP-complete for triangle-free graphs [Maffray et al., 1996] (by this reduction nothing is implied for maximal triangle-free graphs)
- almost all graphs G have $diam(G) \leq 2$ [Bollobás, 1981]

Main results

For graphs G with $diam(G) \leq 2$:

- a very simple, $2^{O(\sqrt{n \log n})}$ subexponential algorithm (worst-case),
- a subclass of graphs with diameter at most 2 that admits a polynomial algorithm
    - locally decomposable graphs.

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- a subclass of graphs with diameter at most 2 that admits a polynomial algorithm
  - locally decomposable graphs.

For graphs G with $diam(G) \leq 3$:

- for every $\varepsilon \in [0, 1)$, 3-coloring is NP-complete for triangle-free graphs G where $diam(G) \leq 3$, $rad(G) \leq 2$, and minimum degree $\delta = \Theta(n^{\varepsilon})$,
- a $2^{O(\min\{\delta\Delta, \frac{n}{\delta} \log \delta\})}$ subexponential algorithm, where $\Delta =$ max. degree,
- subexponential lower bounds (assuming *Exp. Time Hypothesis*):

$\delta(G) = \Theta(n^{\varepsilon})$:	$0 \leq \varepsilon < \frac{1}{3}$	$\frac{1}{3} \leq \varepsilon < \frac{1}{2}$	$\frac{1}{2} \leq \varepsilon < 1$
Lower bound:	no $2^{o(n^{(\frac{1-\varepsilon}{2})})}$-alg.	no $2^{o(n^{\varepsilon})}$-alg.	no $2^{o(n^{1-\varepsilon})}$-alg.

- for $\frac{1}{2} \leq \varepsilon < 1$ this lower bound is almost tight.

Preprocessing of the input graph

- If a graph G has an induced 4-clique then G is not 3-colorable.
- For any input graph G, we apply two reduction rules that do not affect 3-colorability:
  1. Siblings elimination: If $N(u) \subseteq N(v)$ for a pair of vertices u, v, then remove u from G.
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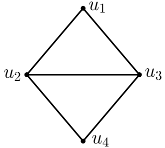
  2. Diamond elimination: If the vertices $\{u_1, u_2, u_3, u_4\}$ induce a diamond where $u_1 u_4 \notin E$, then merge u_1 and u_4.

diamond:



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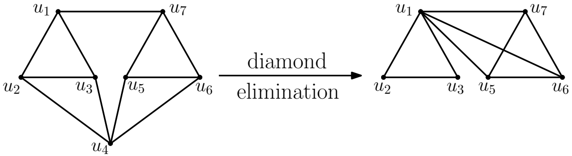

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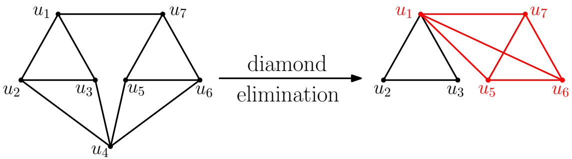

- After a diamond elimination:
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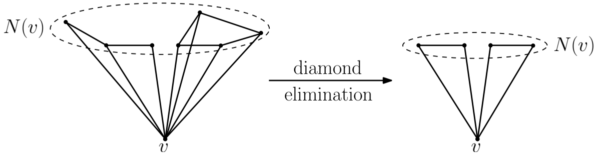

- After a diamond elimination:
  - new 4-cliques may appear!
  - the induced neighborhood of every vertex v has maximum degree 1.

Preprocessing of the input graph

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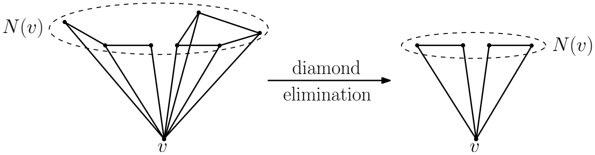

- If these reduction rules do not apply to G, then G is called irreducible.

$\Longrightarrow$ W.l.o.g. the input graph is irreducible and 4-clique-free.

- We use a Dominating Set approach:

Lemma (Narayanaswamy and Subramanian, *Inf. Proc. Let.*, 2011)

Let $G = (V, E)$ be a graph and D be a dominating set of G. Then, the 3-coloring problem can be decided in $O^*(3^{|D|})$ time.

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Let $G = (V, E)$ be a graph and D be a dominating set of G. Then, the 3-coloring problem can be decided in $O^*(3^{|D|})$ time.

Proof (sketch).

- Generate all proper 3-colorings χ_D of the dominating set D (in $O^*(3^{|D|})$ time)
- Every vertex $u \notin D$ has at least one neighbor in D

$\Rightarrow$ every $u \notin D$ has at most two available colors compatible to χ_D

- For every χ_D, solve a list-2-coloring problem for $V \setminus D$ (in polynomial time). □

Theorem

Let G be a graph with $diam(G) \leq 2$ and δ be its minimum degree. Then, the 3-coloring problem can be decided in $2^{O(\min\{\delta, \frac{n}{\delta}\log\delta\})}$ time.

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First Algorithm:

- We construct a dominating set D with $|D| \leq n\frac{1+\ln(\delta+1)}{\delta+1}$ (in polynomial time) [Alon, *Graphs and Combinatorics*, 1990]
- By the previous lemma we decide 3-coloring in time $O^*(3^{n\frac{1+\ln(\delta+1)}{\delta+1}})$, i.e. in time $2^{O(\frac{n}{\delta}\log\delta)}$.

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Proof (sketch).

Second Algorithm:

- Let v be a vertex with $\deg(u) = \delta$.
- Since $diam(G) \leq 2$, $N(u)$ is a dominating set with $|N(u)| = \delta$.
- By the previous lemma we decide 3-coloring in time $2^{O(\delta)}$.

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Summarizing, we decide 3-coloring in time $2^{O(\min\{\delta, \frac{n}{\delta}\log\delta\})}$. □

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Corollary

The 3-coloring problem can be decided in $2^{O(\sqrt{n\log n})}$ time on graphs G with $diam(G) \leq 2$.

Graphs G with $diam(G) \leq 2$

A subexponential algorithm

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Question: Is this indeed the worst case complexity of our algorithm?

i.e.: do there exist 3-colorable, irreducible graphs G with $diam(G) \leq 2$, such that both $\delta(G)$ and the size of the minimum dominating set is $\Theta(\sqrt{n \log n})$?

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Answer: Almost yes.

Theorem

For every $n \geq 1$ there exists an irreducible and triangle-free 3-colorable graph G_n with $diam(G_n) = 2$ and $\Theta(n)$ vertices, where both $\delta(G_n)$ and the size of the minimum dominating set are $\Theta(\sqrt{n})$.

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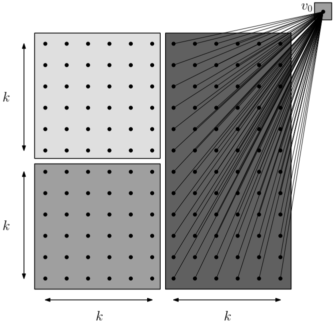
Main idea: arrange the vertices on a matrix $n \times n$



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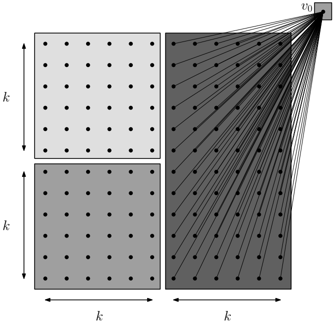

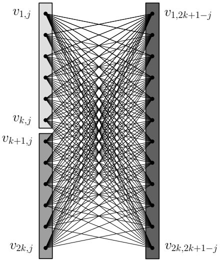

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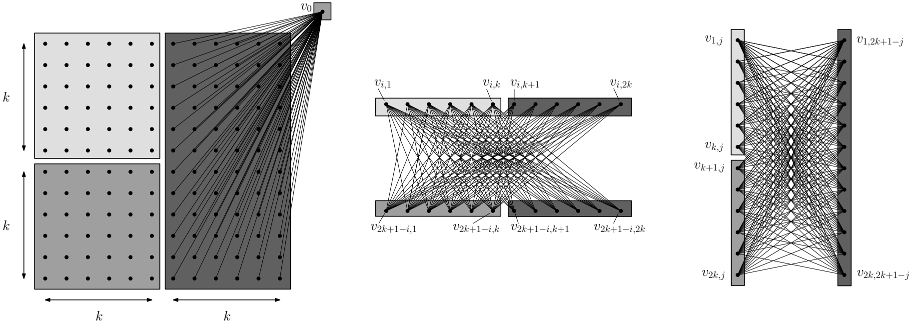


Graphs G with $diam(G) \leq 2$

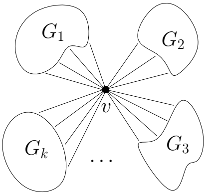
A tractable subclass

- For a vertex v, $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v.

Definition

Let $G = (V, E)$ be a connected graph and $v \in V$. Then

- v is an articulation vertex if $G - \{v\}$ is disconnected,



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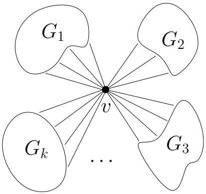

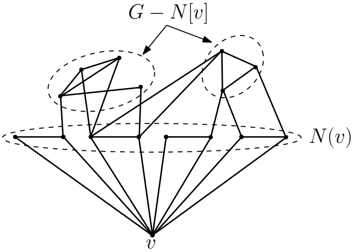

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Definition

A connected graph G is locally decomposable if it has at least one articulation neighborhood.

- Articulation neighborhoods make 3-colorability easy when $diam(G) \leq 2$

Graphs G with $diam(G) \leq 2$

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Theorem

Let $G = (V, E)$ be an irreducible graph with $diam(G) \leq 2$. If G is locally decomposable then 3-colorability can be decided in polynomial time on G.

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Sketch of the algorithm:

- Let $v_0 \in V$ such that $G - N[v_0]$ is disconnected.
- Let $C_1, C_2, \ldots, C_k$, $k \geq 2$, be the components of $G - N[v_0]$.

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- Assume three colors: blue, green, red.
- Color v_0 red $\implies$ all vertices of $N(v_0)$ are colored blue / green.

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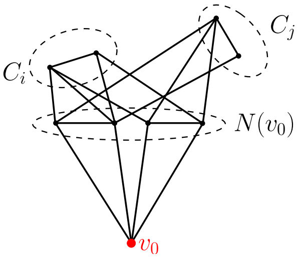
- Assume three colors: blue, green, red.
- Color v_0 red $\implies$ all vertices of $N(v_0)$ are colored blue / green.
- Every C_i has at least one edge:
  - otherwise C_i has only one vertex u where $N(u) \subseteq N(v_0)$, contradiction (by the siblings elimination).

$\implies$ in any coloring, every C_i has at least one blue or green vertex.

Sketch of the algorithm (continued):

- Let C_i, C_j be an arbitrary pair of components.



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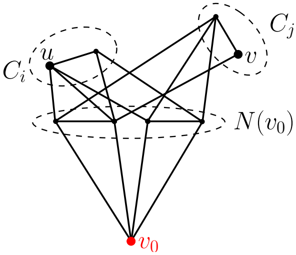

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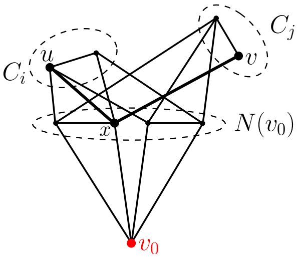
$\Longrightarrow$ u and v must have at least one common neighbor x in $N(v_0)$.



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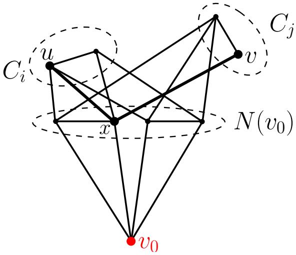

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- All x, u, v are not red $\Longrightarrow$ both u and v have the same color.
- This applies to every pair u, v of non-red vertices in C_i, C_j, where $1 \leq i < j \leq k$

$\Longrightarrow$ $G - N[v_0]$ is bipartite (e.g. with colors red and blue).

Sketch of the algorithm (continued):

- Eligible colors for:
  - every vertex of $G - N[v_0]$ are {red, blue},
  - every vertex of $N(v_0)$ are {blue, green}
  - the vertex v_0 is {red}

$\Longrightarrow$ we solve the list-2-coloring in polynomial time.

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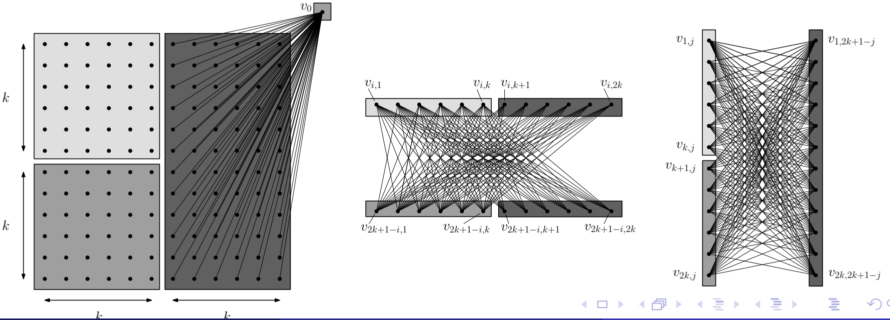
Answer: No.

Graphs G with $diam(G) \leq 2$

A subexponential algorithm

- This graph G is:
  - 3-colorable,
  - irreducible,
  - has $diam(G) = 2$, and
  - has no articulation neighborhood
    (i.e. $G - N[v]$ is connected for every vertex v).

Main idea: arrange the vertices on a matrix $n \times n$



Graphs G with $diam(G) \leq 3$

A subexponential algorithm

Theorem

Let G be a graph with $diam(G) \leq 3$ and δ, Δ be its minimum and maximum degrees. Then, we can decide 3-coloring in $2^{O(\min\{\delta\Delta, \frac{n}{\delta}\log\delta\})}$ time.

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Proof (sketch).

First Algorithm:

- Similarly to the case $diam(G) = 2$, we obtain an $2^{O(\frac{n}{\delta}\log\delta)}$-algorithm.

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Proof (sketch).

Second Algorithm:

- Let v be a vertex with $|N(v)| = \delta$.
- Let A be the vertices with distance 2 from v.
- Every vertex of A is adjacent to a vertex of $N(v) \implies |A| \leq \delta\Delta$
- Since $diam(G) \leq 3$, $A \cup \{v\}$ is a dominating set.
- By the Dominating-Set approach, we decide 3-coloring in time $2^{O(\delta\Delta)}$.

□

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$\Longrightarrow$ Unless $\delta = O(1)$ and $\Delta = \Theta(n)$, we decide 3-coloring in subexponential time for graphs G with $diam(G) \leq 3$.

Graphs G with $diam(G) \leq 3$

NP-completeness

- Reduction from the 3SAT problem
- Given a 3-CNF formula ϕ with n variables and m clauses, construct a graph H_ϕ such that:
  1. $diam(H_\phi) = 3$,
  2. $rad(H_\phi) = 2$,
  3. H_ϕ is irreducible, and
  4. H_ϕ is triangle-free.

  and:

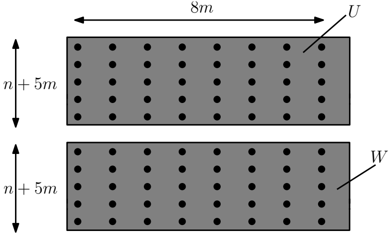
Theorem

H_ϕ is a 3-colorable $\Leftrightarrow$ ϕ is satisfiable.

- Before we construct H_ϕ, we first construct a graph $G_{n,m}$ that depends only on the numbers n and m.

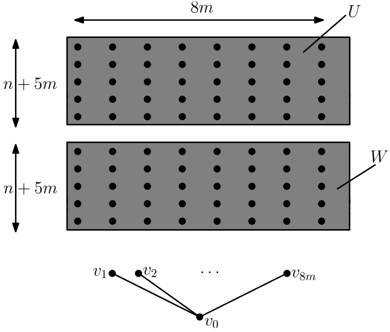
Construction of the graph $G_{n,m}$:

- two matrix arrangements U, W with $(n + 5m) \times 8m$ vertices each,



Construction of the graph $G_{n,m}$:

- two matrix arrangements U, W with $(n+5m) \times 8m$ vertices each,
- a "central" vertex v_0 with neighbors $v_1, v_2, \ldots, v_{8m}$,



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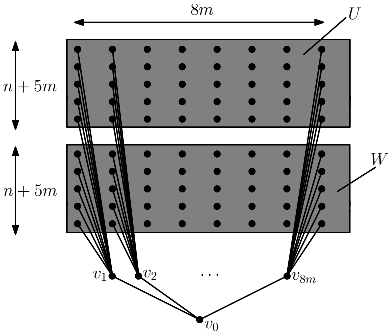
- connect every vertex v_i with the whole ith column of U and W,



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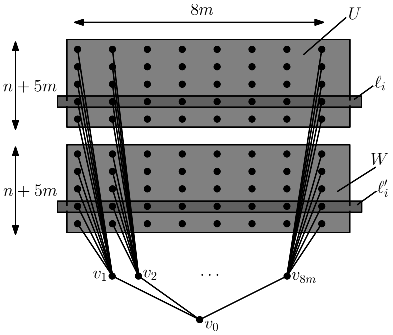
- for the corresponding rows of U and W:
  complete bipartite graph without a perfect matching.



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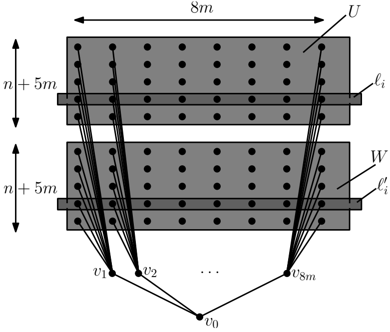
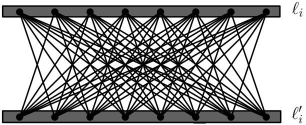
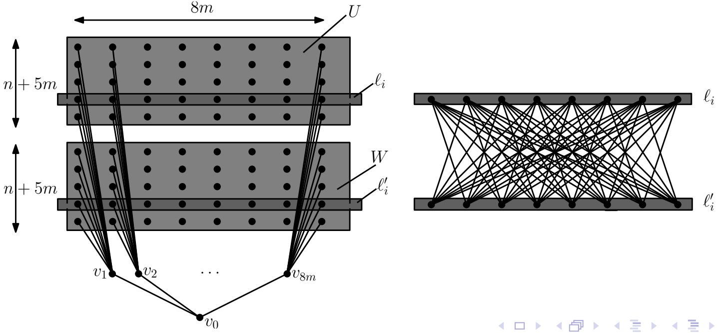




Construction of the graph $G_{n,m}$:

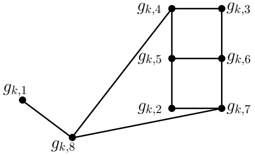
Lemma

For every $n, m \geq 1$, we have $diam(G_{n,m}) = 3$ and $rad(G_{n,m}) = 2$.



Construction of the graph H_ϕ from $G_{n,m}$:

- for every clause $\alpha_k = (\ell_{k,1} \vee \ell_{k,2} \vee \ell_{k,3})$ of ϕ, we add to $G_{n,m}$ an isomorphic copy of this gadget:



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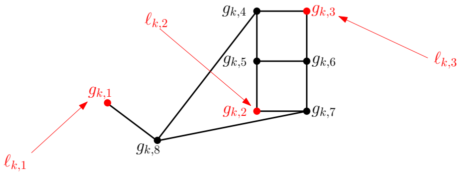

- where the literal $\ell_{k,i}$ corresponds to vertex $g_{k,i}$, where $i = 1, 2, 3$.

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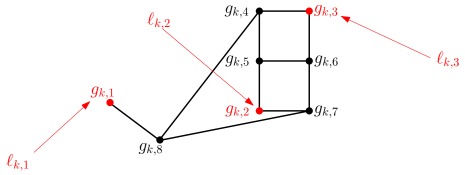

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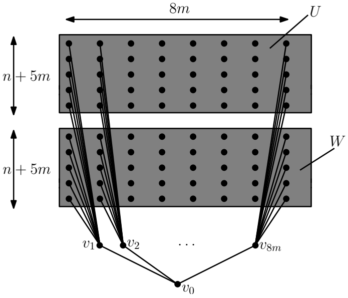
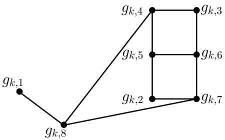
Lemma

- *This gadget is 3-colorable but not 2-colorable.*
- *If all vertices $g_{k,1}, g_{k,2}, g_{k,3}$ receive the same color, then the gadget needs at least 4 colors.*

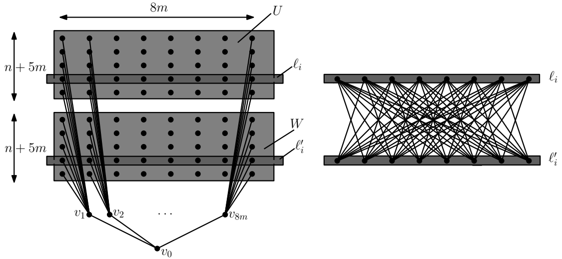
Construction of the graph H_ϕ from $G_{n,m}$:

- We add this gadget on the top of the matrix arrangement of U, W
    - such that every vertex of the m gadget copies lies on a different column (and on the appropriate row of U or W).





- Main idea for the 3-coloring of H_ϕ:
  - color v_0 red and each of the $v_1, \ldots, v_{8m}$ blue or green,
  - for two corresponding rows of U and W, one is entirely red and the other is mixed blue / green
    (this encodes the variable values $x_i \in \{0, 1\}$).



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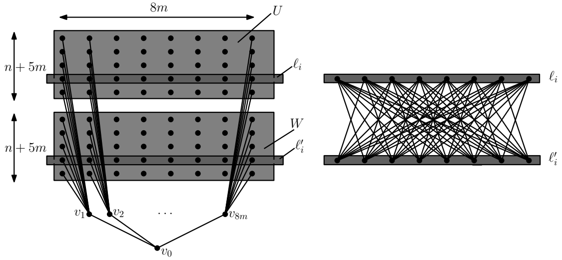
- For the kth gadget copy, not all vertices $\{g_{k,1}, g_{k,2}, g_{k,3}\}$ are red

$\Leftrightarrow$ the clause $\alpha_k = (\ell_{k,1} \vee \ell_{k,2} \vee \ell_{k,3})$ is satisfied



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H_ϕ is a 3-colorable $\Leftrightarrow$ ϕ is satisfiable.

Therefore:

Theorem

3-coloring is NP-complete on irreducible, and triangle-free graphs of diameter 3 and radius 2.

- We can now modify our construction in two ways, in order to amplify our hardness results:

Theorem

Let $\varepsilon \in [0, \frac{1}{2})$. Then 3-coloring is NP-complete on irreducible triangle-free graphs $G = (V, E)$ with $diam(G) = 3$, $rad(G) = 2$, and $\delta(G) = \Theta(|V|^{\varepsilon})$.

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More amplifications, assuming the ETH

Exponential Time Hypothesis (ETH) (Impagliazzo et al., 2001)

There exists no algorithm solving 3SAT in time $2^{o(n)}$, where n is the number of variables in the input CNF formula.

Theorem (Narayanaswamy et al, 2011; Lokshtanov et al., 2011)

Assuming ETH, there exists no $2^{o(n)}$ time algorithm for 3-coloring on graphs G with diameter 4, radius 2, and n vertices.

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- We can now modifying our reduction even more, in order to obtain asymptotic lower bounds for graphs with diameter 3 (assuming ETH):

$\delta(G) = \Theta(n^{\varepsilon})$:	$0 \leq \varepsilon < \frac{1}{3}$	$\frac{1}{3} \leq \varepsilon < \frac{1}{2}$	$\frac{1}{2} \leq \varepsilon < 1$
Lower bound:	no $2^{o\left(n^{\left(\frac{1-\varepsilon}{2}\right)}\right)}$-alg.	no $2^{o(n^{\varepsilon})}$-alg.	no $2^{o(n^{1-\varepsilon})}$-alg.

Recall:

- we provided a $2^{O(\min\{\delta\Delta, \frac{n}{\delta}\log\delta\})}$ time algorithm for graphs G with $diam(G) \leq 3$

$\Longrightarrow$ for $\delta = \Theta(n^{\varepsilon})$, $\frac{1}{2} \leq \varepsilon < 1$, we have a $2^{O(n^{1-\varepsilon}\log n)}$ time algorithm

$\Longrightarrow$ for $\frac{1}{2} \leq \varepsilon < 1$ our lower bound is asymptotically (almost) tight.

$\delta(G) = \Theta(n^{\varepsilon})$:	$0 \leq \varepsilon < \frac{1}{3}$	$\frac{1}{3} \leq \varepsilon < \frac{1}{2}$	$\frac{1}{2} \leq \varepsilon < 1$
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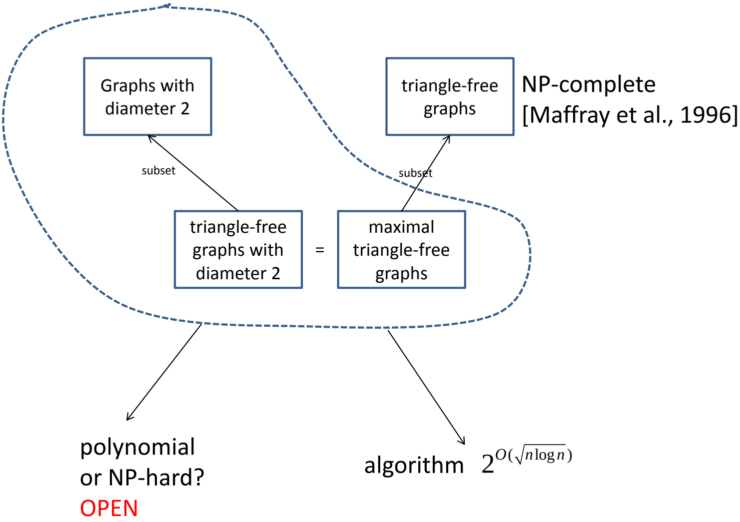
Summary and open problems

- Our results for 3-coloring on graphs with diameter at most 2:
  - a simple $2^{O(\sqrt{n \log n})}$ time algorithm (worst case),
  - a polynomial algorithm for a subclass (locally decomposable graphs).
- Our results for 3-coloring on graphs with diameter at most 3:
  - a $2^{O(\min\{\delta\Delta, \frac{n}{\delta} \log \delta\})}$ time algorithm,
  - NP-hardness for graphs with minimum degree $\delta = \Theta(n^{\varepsilon})$, $\varepsilon \in [0, 1)$,
  - subexponential asymptotic lower bounds (assuming ETH), parameterized by the minimum degree.
- Can 3-coloring be solved polynomially on graphs with diameter at most 2?
- What happens if the graph is also triangle-free?
- For graphs G with diameter at most 3, can we find matching asymptotic lower bounds for every $\delta(G)$?

Summary and open problems



Thank you for your attention!