A natural counting of lambda terms

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joint work with
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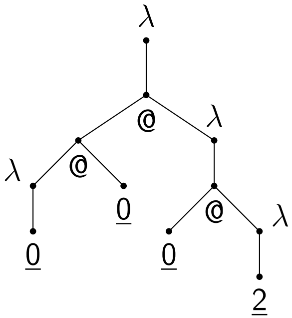
Motivations

1. Combinatorics – design new methods for counting structures with binders and local scopes,
2. Computer Science – develop tools for random λ-term generation used in software testing (see, e.g. Quickcheck),
3. Computational Logic – study quantitative aspects of semantic properties in λ-calculus and related systems,
4. . . .

Natural size notion



$$\lambda x.((\lambda y.y)x)(\lambda z.z(\lambda w.x))$$

$$\lambda((\lambda\underline{0})\underline{0})(\lambda\underline{0}(\lambda\underline{2}))$$

$$|t| = 13$$

Natural counting of λ-terms

$$\mathcal{L}_\infty = \mathcal{L}_\infty \, \mathcal{L}_\infty \;\oplus\; \lambda \, \mathcal{L}_\infty \;\oplus\; \mathcal{D}$$

$$\mathcal{D} = S \, \mathcal{D} \oplus \; \mathtt{0}$$

$$\downarrow$$

$$L_\infty(z) = z L_\infty^2(z) + z L_\infty(z) + \frac{z}{1-z}$$

$$L_\infty(z) = \frac{(1-z)^{3/2} - \sqrt{1 - 3z - z^2 - z^3}}{2z\sqrt{1-z}}$$

Asymptotic approximation of $([z^n]L_\infty)_{n\in\mathbb{N}}$

Theorem (B, Grygiel, Lescanne, Zaionc)

The asymptotic approximation of the number of λ-terms of size n is given by

$$[z^n]L_\infty \sim (3.38298\ldots)^n \frac{C}{n^{3/2}}, \quad \text{where} \quad C \doteq 0.60676.$$

Holonomic presentation of L_∞

$$L_\infty(z) = \frac{(1-z)^{3/2} - \sqrt{1-3z-z^2-z^3}}{2z\sqrt{1-z}}$$

$$\downarrow$$

Maple: package gfun

$$z^3+z^2-2z+(z^3+3z^2-3z+1)L_\infty+(z^5+2z^3-4z^2+z)L'_\infty = 0.$$

Computing $L_{\infty,n}$

'Holonomic' recursion for $L_{\infty,n}$

$$L_{\infty,0} = 0, \qquad L_{\infty,1} = 1, \qquad L_{\infty,2} = 2, \qquad L_{\infty,3} = 4,$$

$$\begin{aligned}(n+1)L_{\infty,n} = \quad & (4n-1)L_{\infty,n-1} - (2n-1)L_{\infty,n-2} \\ & -L_{\infty,n-3} - (n-4)L_{\infty,n-4}.\end{aligned}$$

λ-terms with bounded number of free indices

$$\mathcal{L}_m = \mathcal{L}_m \, \mathcal{L}_m \oplus \lambda \, \mathcal{L}_{m+1} \oplus \mathcal{D}_m$$

$$\mathcal{D}_m = \{\underline{0}, \underline{1}, \ldots, \underline{\text{m-1}}\}$$

$$\downarrow$$

$$L_m(z) = \frac{1 - \sqrt{1 - 4z^2 \left(L_{m+1}(z) + \frac{1-z^m}{1-z} \right)}}{2z}$$

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L_m is expressed by means of infinitely nested radicals!

Counting λ-terms containing fixed subterms

Theorem (B, Grygiel, Lescanne, Zaionc)

For a fixed term M, the asymptotic density of $\mathcal{T}_{\mathsf{M}}$ is equal to 1. In other words, asymptotically almost all λ-terms contain M as a subterm.

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Proof sketch

1. $\mathcal{T}_{\mathsf{M}} = \mathsf{M} \oplus \lambda\, \mathcal{T}_{\mathsf{M}} \oplus \mathcal{T}_{\mathsf{M}}\, \mathcal{L}_\infty \oplus \mathcal{L}_\infty\, \mathcal{T}_{\mathsf{M}} \ominus \mathcal{T}_{\mathsf{M}}\, \mathcal{T}_{\mathsf{M}}$.
2. Consider $L_\infty(z) - T_{\mathsf{M}}(z)$. Show that it has density 0.

Counting λ-terms containing fixed subterms - cnd.

Theorem (B, Grygiel, Lescanne, Zaionc)

Asymptotically almost no λ-term is strongly normalizing.

The sequence $([z^n]L_\infty)_{n\in\mathbb{N}}$

The sequence $([z^n]L_\infty)_{n\in\mathbb{N}}$ is known as **A105633** in *Online Encyclopedia of Integer Sequences* (`http://oeis.org`)!

0, 1, 2, 4, 9, 22, 57, 154, 429, 1223, 3550, 10455, 31160, 93802, 284789, 871008, 2681019, 8298933, 25817396,. . .

E-free black-white binary trees

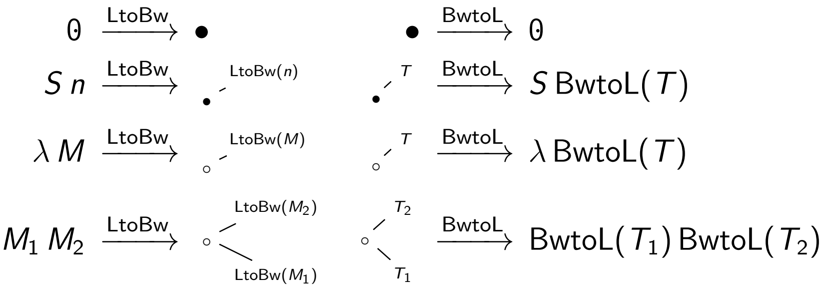
Black-white binary trees (A105633)

1. $A_1 = \{ {}_{\circ}\diagup^{\bullet}, {}_{\bullet}\diagup^{\bullet}, {}_{\circ}\diagup^{\circ}, {}^{\circ}\diagdown_{\bullet} \}$,
2. Roots are black.

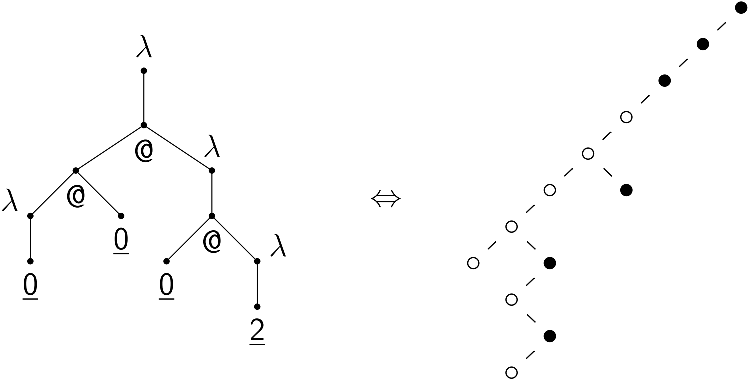
$$\downarrow$$

$$BW_{\bullet}(z) = z + zBW_{\bullet}(z) + zBW_{\circ}(z)$$
$$BW_{\circ}(z) = z + zBW_{\circ}(z) + zBW_{\bullet}(z) + zBW_{\circ}(z)BW_{\bullet}(z)$$

Bijection between λ-terms and black-white trees

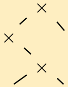


Example



Binary trees without zigzags

Zigzag-free binary trees (**A105633**)



$\downarrow$

$$\mathcal{BZ}_1 = \begin{array}{cc}\times & \\ & \diagdown \\ & \mathcal{BZ}_1\end{array} \oplus \mathcal{BZ}_2$$

$$\mathcal{BZ}_2 = \times \oplus \begin{array}{cc} & \times \\ \diagup & \\ \mathcal{BZ}_2 & \end{array} \oplus \begin{array}{ccc} & \times & \\ \diagup & & \diagdown \\ \mathcal{BZ}_2 & & \mathcal{BZ}_1\end{array}$$

Binary trees without zigzags - continued

Theorem (B, Grygiel, Lescanne, Zaionc)

There exists a computable (linear) bijection between black-white and zigzag-free trees and thus between λ-terms and zigzag-free trees.

Thank you!

Questions & Answers